CHAPTER

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Boolean Logic in Fluid Flow

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OUTLINE

In learning how to use Boolean algebra concepts in geohydrology, it would be easier for students to start with examples and then bring in the theory, as is practiced by many instructors. In the case of a current flow, we can think of a switch, which if "on" (Fig. 1A), the current would pass, and if "off" (Fig. 1A), it does not. The switch-on case can be denoted by "1" (unit element, the lamp emits light) and off by "0" (zero element, the lamp does not emit light). The following *truth table* can be constructed to express whether the state of lamp is on or off when the two switches, x and y, are *in a series network,* also called *"and circuit"/"and gate"* or an *"xy circuit"*, represented by a dot symbol (·). This dot does not mean multiplication in the sense of arithmetic (Fig. 1C).

Switch x	Switch y	State of lamp
0	0	0
0	1	0
1	0	0
1	1	1

On the other hand, a *parallel network* of switches called *"or circuit"/"or gate"* or "$(x+y)$ *circuit"*, represented by a plus (+) symbol will have the following truth table (Fig. 1D):

Switch x	Switch y	State of lamp
0	0	0
0	1	1
1	1	1
1	0	1

Thus the plus symbol here does not mean arithmetic addition. One can represent x as "flow taking place" and $x^/$ as "no flow taking place". $x^/$ is called the complement of x. In a logic gate, this is called the "*not gate*". Therefore using the plus symbol as defined earlier here,

$$x + x^/ = x^/ + x = 1 \quad (1)$$

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Also note

$$\left(x^{/}\right)^{/}=x \tag{2}$$

We can now similarly bring up the concept of fluid flow through permeable geological media. Refer to Fig 3.2.2 in Todd and Mays (2012) for ranges of permeability values for the common rock types. Fig. 1C can have an equivalent case of Fig. 1C$^{/}$. Here a permeable formation and an impermeable formation are in a series connection. Similarly, Fig. 1D$^{/}$ represents a parallel connection between two such layers.

We will now see some (easy) combination of $x+y$ and xy circuits. Fig. 2A$^{/}$ shows an impermeable layer (shown by hatches) at the bottom and a permeable layer (shown by dots) above it. Further, due to facies variation, there is another impermeable unit deposited within the permeable layer. Can a fluid pass from left to right, or vice versa, across these permeable and impermeable sediments? The common sense answer is "no". Let's draw the equivalent diagram for current flow case, Fig. 2A. Using symbols,

$$\left(x.x^{/}.x\right)+x^{/}=0 \tag{3}$$

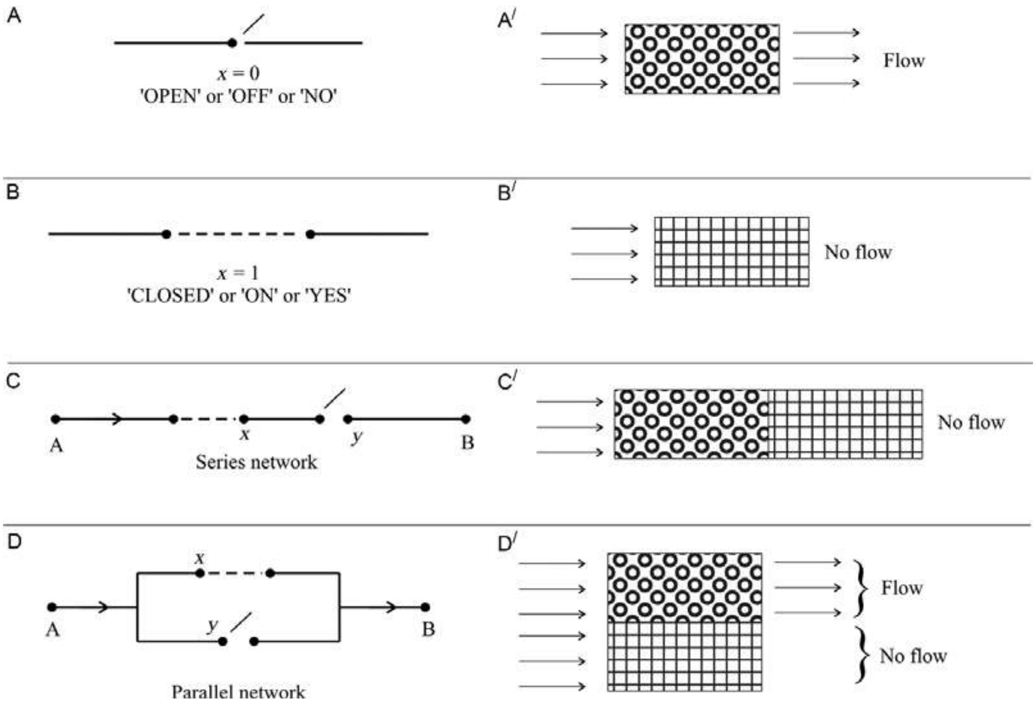

FIG. 1 (A) A switch, in the current flow context, that is "off". (B) A switch, in the current flow context, that is "on". (C) A permeable and an impermeable formation in series connection. (D) A permeable and an impermeable formation in parallel connection.

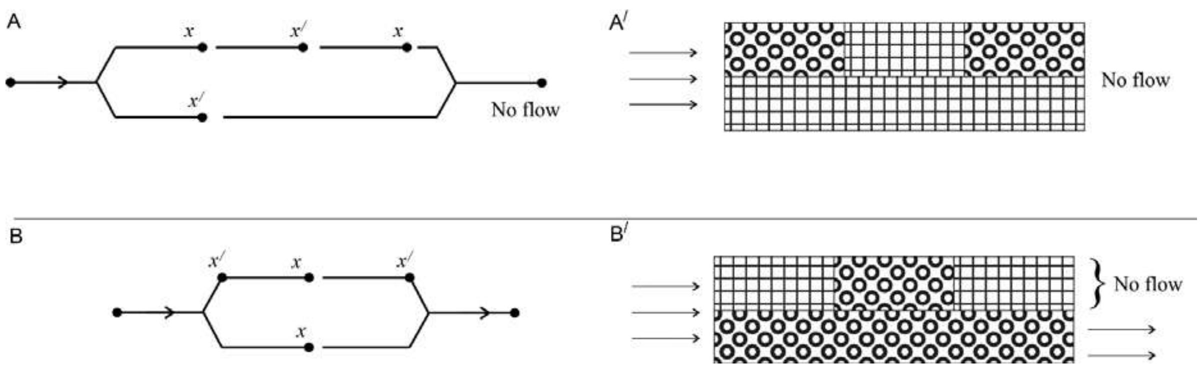

FIG. 2 (A–C) Combined disposition of permeable and impermeable geological formations. (A) An electrical circuit equivalent to the case 2A$^{/}$.

Here $(x.x^{/}.x)$ represents the flow situation inside the top permeable layer, inside which a patch of impermeable layer exists. Individually,

$$\left(x.x^{/}.x\right)=0, \text{ or "no flow from left to right or right to left".} \quad (4)$$

Putting (4) in (3):

$$0+x^{/}=0 \quad (5)$$

Question 1: For the disposition if permeable and impermeable sediments as in Fig. 2B$^{/}$, write down its corresponding equation. Consider "x" as a permeable unit allowing flow and "$x^{/}$" as impermeable unit precluding flow.

Question 2: A more complicated yet easily solvable case! Write down the equation for Fig. 3A case.

Let's now see the circuit simplification case discussed in Boolean algebra, and how the problem is translated to hydrology. If all the switches are either all open or are all closed, they are represented by say "x". On the other hand, if for another set of switches, some are open and some closed, let them be represented by "y". Any third set of switches may be represented by "z". For a circuit as in Fig. 4A, the following table can be constructed where "1" represents on and "0" off.

Switch x	Switch y	State of lamp
0	0	0
0	1	1
1	0	0
1	1	1

Note the corresponding elements in the previous matrix in the second and the third column are exactly the same. This means when switch y is on, the lamp lights, and when it is off, it does not. Whether or not switch x is present does not control the illumination of the lamp. Therefore Fig. 4A$^{/}$ would be the simplified or the effective circuit in this case. Comparing Fig. 4A and A$^{/}$, we can state that their respective Boolean expressions for circuits are basically the same, or,

$$y+x.y=y \quad (6)$$

Eq. (6) is the "absorption law." Fig. 5A and A$^{/}$ are the corresponding hydrological cases.

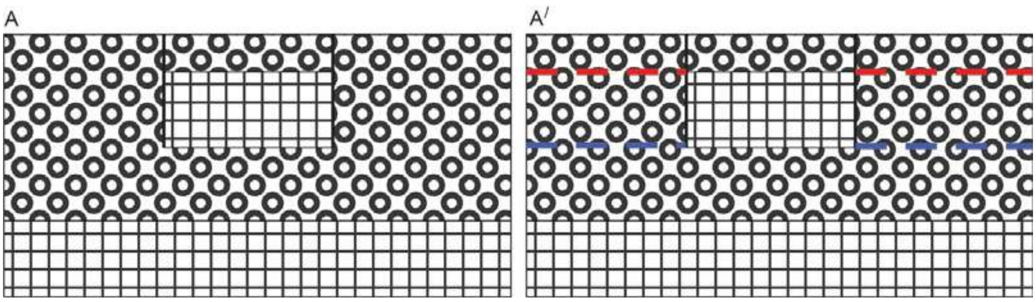

FIG. 3 (A) Permeable and impermeable unit disposition. (B) To deduce the Boolean algebraic equation, few additional dash lines.

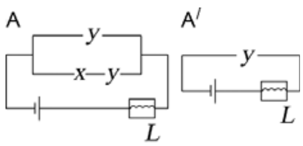

FIG. 4 (A) Electric circuit and its simplification in B.

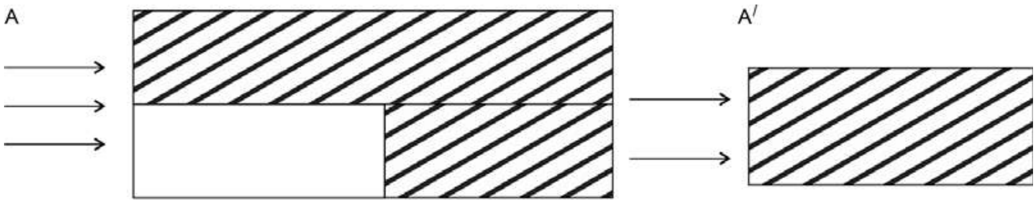

FIG. 5 (A) Permeable and impermeable unit disposition, similar to Fig. 4A. (B) Simplification of the case of A, and this is similar to Fig. 4B.

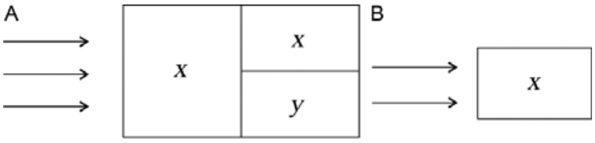

FIG. 6 (A) Permeable (represented by 'x') and impermeable (represented by 'y') layer disposition. (B) Its simplification.

Question 3: What would be the simplification of the hydrological case: $x.(x+y)$, as shown in Fig. 6A?
The following equations can be constructed easily:

$$(x+y)^{/} = x^{/}.y^{/} \tag{7}$$

$$(x.y)^{/} = x^{/} + y^{/} \tag{8}$$

$$x + x^{/} = 1 \tag{9}$$

$$x.x^{/} = 0 \tag{10}$$

$$1^{/} = 0 \tag{11}$$

$$0^{/} = 1 \tag{12}$$

In terms of a logic gate, Eq. (7) can also be called a *"nand gate,"* as two operations work simultaneously: first "and," given by $(x+y)$; and then it is the "not" operation given by the complement symbol ($^{/}$). Eq. (8) can also be called a "*nor gate*," as two operations work simultaneously: first "or," given by $(x.y)$; and then it is the "not" operation given by the complement symbol ($^{/}$).

Let's study Eq. (7) in detail. (1) If both x and y indicate permeable units, $(x+y)$ indicate flow, therefore $(x+y)^{/}$ means "no flow." Let's focus now at the right hand side of the equation: both $x^{/}$ and $y^{/}$ mean impermeable units, so the right side of the equation also indicates no flow, matching with the left side of the equation. (2) If x and y indicate permeable and impermeable units, respectively, then their parallel circuit indicates flow. Therefore $(x+y)^{/}$ means "no flow." Let's look at now the right hand side of the equation. It depicts a series connection between an impermeable ($x^{/}$) and a permeable unit (y). Together, in a series, this means no flow. (3) A similar proof as in (2) will work if x and y are impermeable and a permeable unit, respectively. (4) Consider now both x and y are impermeable units. Then $(x+y)$ would mean no flow, hence $(x+y)^{/}$ would mean flow taking place. Lets focus now the right hand side of the equation. $x^{/}$ and $y^{/}$ would mean permeable units and therefore $x^{/}.y^{/}$ connotes flow taking place. This matches with the left hand side of the equation. Eqs. 8-10 can be understood in the same way.

Temporal variation of permeability would be possible by fracturing, for example, caused by tectonic stresses, or fracture and pore-space filling by secondary deposition. Thus its flow behavior can change with time from "1" to "0" or vice versa, or from x to $x^{/}$.

Likewise, one can verify the following in a *state table*, for any flow:

a	b	a'	b'	$(a+b)$	$(a+b)'$	$a'.b'$
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	0	1	1	0	1	1
0	0	1	1	0	1	1

Lets construct the following state table for the case of Fig. 7.

Layer x	**Layer y**	**Fluid flow (1: yes; 0: no)**
1	1	1
1	0	1
0	0	0
0	1	1

Though not stated in the previous table, you obviously need to also think of:

Layer x	**Layer $x^{/}$**
1	0
0	1

The flow can be represented by: $(x+x^{/}.y)$. Interestingly, one can construct the state table for $(x+y)$ and that matches exactly with the previous one. Therefore,

$$\left(x+x^{/}.y\right)=(x+y) \tag{13}$$

One can consider graph theory (Mukherjee, in press) and Boolean logic together. For example, in Fig. 8A, fluid flow path is shown in Fig. 8B amongst several vertices (v_i) through edges (e_i). If a flow starts from V_i vertex and reaches $V_i^{/}$ vertex through the edge e_i, then the element at the (V_i, e_i) element in the matrix is to be "1", and that for $(V_i^{/}, e_i)$ is to be "−1". This is represented by the following matrix.

	e_1	e_2	e_3	e_4
v_1	+1	0	0	0
v_2	0	0	0	0
v_3	−1	+1	+1	0
v_4	0	0	−1	0
v_5	0	−1	0	+1
v_6	0	0	0	−1

From v_1 to v_2 no flow is possible because the imaginary line joining them passes through two impermeable zones. No edge is therefore established between v_1 and v_2. The connections amongst v_i can be represented by:

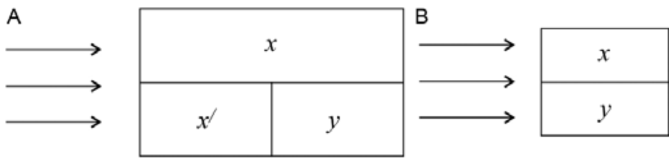

FIG. 7 (A) Permeable and impermeable layer disposition. If "x" represents permeable layer, $x^{/}$, means impermeable unit, and vice versa. "y" can be either permeable or impermeable. (B) Simplification of (A) case.

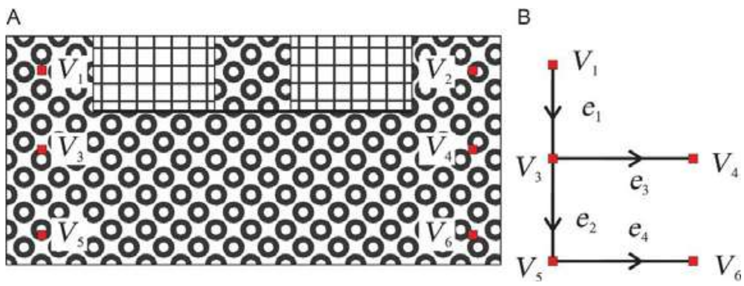

FIG. 8 (A) Permeable and impermeable layer disposition. (B) Fluid flow paths (edges: e_i) through several points/ nodes/vertices: v_i. Note not all vertices are connected by edges.

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	1	0	0	0
v_2	0	0	0	1	0	0
v_3	1	0	0	1	1	0
v_4	0	1	1	0	0	1
v_5	0	0	1	0	0	1
v_6	0	0	0	1	1	0

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APPENDIX

Answer 1: The equation is:

$$\left(x^{/}.x.x^{/}\right)+x=1 \quad \text{(flow takes place)} \tag{i}$$

$$\text{Also note } \left(x^{/}.x.x^{/}\right)=0 \quad \text{(no flow)} \tag{ii}$$

Putting (ii) into (i),

$$0+x=1 \tag{iii}$$

Answer 2: Referring to Fig. 3A$^{/}$, the equation is:

$$x^{/}+x^{/}.x.x^{/}+x^{/}+x=1 \qquad \text{(here } x \text{ represents permeable unit)} \tag{iv}$$

Answer 3: Fig. 6A$^{/}$:

$$x.(x+y)=x \tag{v}$$

Reason: If x is impermeable layer permitting no flow, then no flow is possible throughout the system. If x is a permeable layer permitting flow, an overall flow is possible whether or not y permits flow. So the permeability of x decides effectively whether flow takes place or not.

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Further Reading

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