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Statistical analysis of DWT coefficients of fGn processes using ARFIMA(p,d,q) models

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ABSTRACT

Fractional Gaussian noise (fGn) provides an important parametric representation for the data recorded from long-memory processes. Also it has been well established in literature that the orthogonal wavelet transforms prove to be the optimal bases to represent the data as fGn or fBm (fractional Brownian motion) models. This paper highlights the statistical properties of discrete wavelet transform (DWT) coefficients in the wavelet expansion of fGn. Statistical analysis was carried out by analyzing the inter-scale and intra-scale correlations of the DWT coefficients for wavelets with varying vanishing moments. Two types of auto-regressive moving average (ARMA) models were fit to the wavelet coefficients of fGn, namely, (i) ARMA(p,q) and (ii) ARFIMA(p,d,q) models. The latter represents the ARMA models with fractional differencing. Using the Akaike information criteria (AIC) and the Bayesian information criteria (BIC), it has been shown that ARFIMA models best represent the wavelet coefficients of fGn. The above observation holds good, when wavelets with increasing number of vanishing moments are used for obtaining DWT coefficients. After estimating the optimal model and its parameters, different properties pertaining to the inter-scale and intra-scale correlations were verified using these models.

1. Introduction

Data recorded from a physical or a geophysical system often exhibits a very slow decay of autocorrelation coefficients as the function of lags. As a result, such processes are termed as long-memory processes. The term long-memory refers to the fact that the past values of the data continue to affect the present values [1,2]. The decay of the autocorrelation function is hyperbolic in nature. This is in contrast to the exponential decay of processes, which are modeled as general auto-regressive moving average (ARMA) models. This phenomena was noticed by Hürst [3,4,5], Mandelbrot and Van Ness [6], Mandelbrot [7] and McLeod and Hipel [8]. In geophysics, long-memory processes are observed in a variety of data sets such as, well-log data [9–11], geomagnetic data [12,13] and ionospheric data, [14,15] to mention a few.

Self-similar processes give a parametric representation for long-memory processes. A process $X(t)$ is called self-similar if $\forall\lambda > 0, \quad \lambda \in \mathbb{R}$, we have $X(t) \overset{D}{\sim} \lambda^{-H}X(\lambda t + c)$. Here, $\overset{D}{\sim}$ represents the similarity in finite dimensional distribution [6,16–18]. Self-similar processes are characterized by Hürst parameter (H). The value of Hürst parameter is always between 0–1. If $0 < H < 0.5$, then the process is characterized by short range correlations (i.e, the values in the data fluctuate very fast around a mean value). If $0.5 < H < 1$, then the values taken by the data show persistent behavior.

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In this situation, the increasing trends in the past are followed by the increasing trends in the future and vice versa. Data following such a behavior is said to have long-range dependence. If $H = 0.5$, then the data is said to possess uncorrelated trends. Fractional Brownian motion (fBm) is an example of the stochastic process, having self-similar property. These are zero mean, non-stationary Gaussian random functions [6,19]. The increments of fBm are zero-mean, wide-sense stationary and self-similar and are called fractional Gaussian noise(fGn). If $B_H(t)$ denotes the sample path of the fBm with Hürst parameter H, then the corresponding fGn is given by

$$G_H(t) = \lim_{\Delta\to 0}(B_H(t+\Delta) - B_H(t)) \tag{1}$$

Here, the convergence of the limit is in mean-squared sense [20]. For large lags $\tau, \quad |\tau| \gg 0$, the auto-covariance function of fGn with H is $\mathbb{E}[G_H(t+\tau)G_H(t)] \sim \sigma^2 H(2H-1)|\tau|^{2H-2}$. Power spectral density of fGn, $S_{G_H}(\omega)$, also exhibits power law relation, $S_{G_H}(\omega) \sim |\omega|^{1-2H}$. These two relations were used as a basis for the estimation of H before the advent of wavelet transforms [16,21].

The decay of autocorrelation function for long-memory process is hyperbolic in nature. This decay is slower than the exponential decay of autocorrelation function in case of a general ARMA(p,q) process [22]. Therefore, ARMA(p,q) models cannot incorporate long-range correlations, unless one uses large number of lags to model the data, in which case, the resulting model has a lot of free parameters to be optimized. Granger and Joyeux [23] introduced a fractional differencing operator, which when applied to a white noise sequence, resulted in a process with hyperbolically decaying auto-correlation function, which is a characteristic of long-memory processes [23–26]. ARMA(p,q) models, after introducing fractional differencing are referred to as ARFIMA(p,d,q) models. Here, p and q designate the orders of autoregression and moving average filters and d represents the fractional difference parameter. Beran et al. [27] showed that the ARFIMA (p,d,q) models are wide-sense stationary and self-similar for $|d| < 0.5$. This differencing parameter is related to H by $H = d + \frac{1}{2}$ [27–30]. Parameters of an ARFIMA(p,d,q) models can be jointly estimated in two ways: (i) direct time domain maximum likelihood (ML) estimation [25,31] and (ii) frequency domain approximate ML estimation [32]. Apart from these, one can also follow a two-step estimation procedure for the parameter estimation of ARFIMA(p,d,q) processes [31]. In the first step the differencing parameter is estimated using various time domain estimators [3,33,34], frequency domain estimators [35,36] or wavelet based estimators [37–39]. Using the estimated value of differencing parameter, the fractional differencing is applied to the given data and then the moving average and autoregressive coefficients are estimated using Box–Jenkins approach [22].

Wornell and Oppenheim [40] used the discrete wavelet transform (DWT) [41] for parameter estimation of $1/f$ processes, embedded in white background noise. Wornell [42] also showed that the orthonormal wavelet bases [43] act as optimal bases for representation of long-memory processes, modeled as fGn. Tewfik and Kim [44] studied the correlation structure of DWT coefficients corresponding to a continuous-time fGn and provided the bounds for the decay of autocorrelation function of the wavelet coefficients with increasing number of vanishing moments. Kaplan and Kuo [45] derived an expression for inter-scale (and intra-scale) correlations (and autocorrelations) of Haar bases coefficients corresponding to a discrete fGn (dfGn) and conjectured the upper bounds for the decay of these correlations for wavelet bases, with higher number of vanishing moments. Therefore, the exact expression for correlation (and autocorrelation) function of the DWT coefficients corresponding to discrete fBm (dfBm) and dfGn is not known in the literature. This motivates one to analyze the types of statistical models that best describe these DWT coefficients of the data modeled as fGn.

In this paper, the autocorrelation functions of DWT coefficients, corresponding to different fGns were analyzed. This analysis was done by using wavelets with varying vanishing moments. Since most of the data recorded in physics, geophysics and other allied fields are discrete in nature, the dfBm and dfGn were considered in the analysis. Then, the analysis was further carried out by fitting different statistical models to the DWT coefficients of fGn data. These models are general ARFIMA(p,d,q) models, where the values of the autoregression lag(p) and the order of moving average (q) is varied between (0–3). Also, two types of models were analyzed, one with fractional integration (ARFIMA(p,d,q) models) and one without fractional integration (ARMA(p,q) models). Therefore, a total of 32 different models were analyzed for a given fGn series. Akaike information criteria (AIC) [46] and Bayesian information criteria (BIC) [47] were then used to show that models with fractional integration (models for which $d \neq 0, d \notin \mathbb{Z}$) are better representations for the DWT coefficients of the long-memory processes modeled as fGn. AIC and BIC are very popular model selection criteria, used to assess the relative quality of statistical models. Given a set of candidate models for the data, one can compute AIC and BIC using

$$AIC = 2k - 2\log(\mathcal{L}) \tag{2}$$

$$BIC = \log(N)k - 2\log(\mathcal{L}) \tag{3}$$

where, k is the number of free parameters in the model, $\mathcal{L}$ is the value of the likelihood function [46,47] and N is the number of observations or sample size of the data.

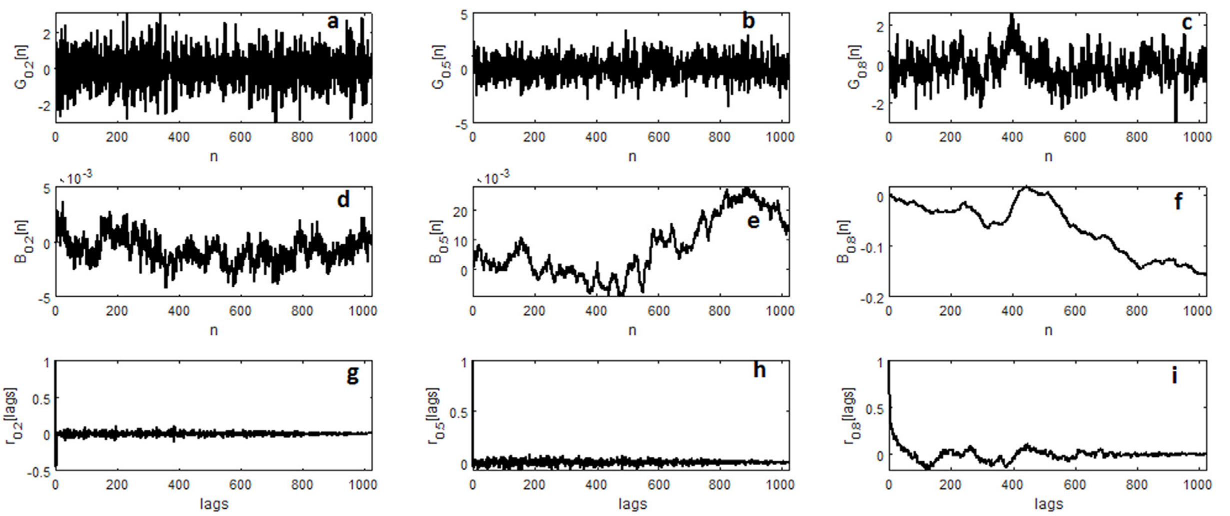

Fig. 1. Sample path of fGn and fBm along with corresponding autocorrelation function for Hürst parameter $H = 0.2$ (a, d, g), $H = 0.5$ (b, e, h) and $H = 0.8$ (c, f, i).

2. Long-memory processes and their wavelet transforms

2.1. Long-memory processes as dfBm/dfGn and ARFIMA(p,d,q) models

Mandelbrot et al. [6,7] showed that the processes in which the correlations persist, can be modeled as fBm. The Covariance of $B_H(t)$ (see Eq. (1)) is given by

$$\mathbb{E}[B_H(t)B_H(t+s)] = r_H(s,t) = \frac{\sigma^2}{2}\left[|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right] \tag{4}$$

In most practical situations, since the data collected is discrete in nature, the dfBm is important to analyze. Sampled version of fBm is denoted as $B_H[n] := B_H(n\Delta t)$, where Δt is the sampling interval. The increments of dfBm, $G_H[n] := B_H[n+1] - B_H[n]$, is a zero-mean stationary Gaussian sequence, known as dfGn [18,48]. They are characterized by their autocorrelation function.

$$r_H[k] := \mathbb{E}[G_H[n+k]G_H[n]] = \frac{\sigma^2}{2}\left[|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}\right] \tag{5}$$

Fig. 1 shows a sample path of fGn (Fig. 1a, Fig. 1b, Fig. 1c) and corresponding fBm (Fig. 1d, Fig. 1e, Fig. 1f). One can see that the autocorrelation function of fGn corresponding to $H = 0.8$ (Fig. 1i) decays slowly compared to $H = 0.5$ (Fig. 1h) and $H = 0.2$ (Fig. 1g) therefore, the fGn corresponding to $H = 0.8$ (or $H > 0.5$) is called long-range dependent process. Another important statistical model for data generated by a long-memory process is that of fractionally integrated autoregressive moving average (ARFIMA) time-series models. Granger et al. [23] showed that, if one adds a fractional differencing operator [24] to a general ARMA(p,q) process, then the decay of autocorrelation as a function of lags follows a hyperbolic decay. This is more gradual than the geometric decay, as in the case of ARMA(p,q) processes. A general ARMA(p,q) can be written as

$$\Phi(B)X[n] = \Theta(B)\epsilon[n] \tag{6}$$

Here, $X[n]$ represents the observation from the process at the time instant n. The backward shift operator B, is defined as $(B^m X)[n] := X[n-m], \quad \forall n, m \in \mathbb{Z}$. $\Phi(B)$ and $\Theta(B)$ are polynomials in B and are defined as

$$\Phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i \tag{7}$$

$$\Theta(B) = 1 - \sum_{i=1}^{q} \theta_i B^i \tag{8}$$

The sequence $\epsilon[n]$, is called innovation sequence or residuals and is a sequence of IID (independent and identically distributed) Gaussian random variables with zero mean and variance σ^2, i.e., $\epsilon[n] \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$. For values of the lags

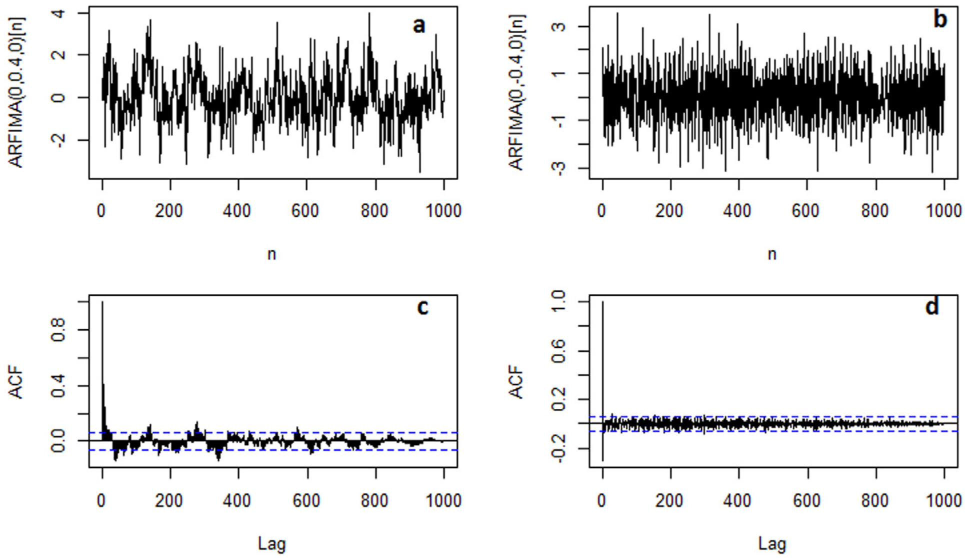

Fig. 2. Sample path of fractionally integrated white noise or ARFIMA(0,d,0) processes along with their autocorrelation functions for ARFIMA(0,0.4,0) (a,c) and ARFIMA(0,−0.4,0) (b,d) processes.

$k, \quad k > q$ the autocorrelation function is given by the recursive equation [22]

$$r_{ARMA}[k] = \sum_{i=1}^{p} \phi_i r_{ARMA}[k-i], \quad k > q \tag{9}$$

General solution to the above difference equation is of the form

$$r_{ARMA}[k] = \sum_{i=1}^{p} c_i z_i^{-k} \tag{10}$$

The constants c_i are obtained via Yule–Walker equation and the constants z_i are roots of the complementary functions associated with the difference equation (9) for the stationary ARMA(p,q) models, $|z_i| < 1$ [22]. As can be seen from Eq. (10) the decay of autocorrelation function is geometric in nature. If one adds the fractional differencing operator in Eq. (6), then the decay of autocorrelation function becomes hyperbolic [31]. Hence, a general fractional differencing operator is defined as [23]:

$$\nabla^d := (1-B)^d = \sum_{j=0}^{\infty} \pi_j B^j \tag{11}$$

Here, $\pi_0 = 1$ and for $j \geq 1$,

$$\pi_j = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)} = \prod_{0<k\leq j} \frac{k-1-d}{k} \tag{12}$$

$\Gamma(x)$ is the gamma function. Also from Eq. (12), one can notice the recursion, $\pi_j = \frac{j-1-d}{j}\pi_{j-1}$. So, the general ARFIMA(p,d,q), for $d \in \mathbb{R}$ is given by

$$\Phi(B)\nabla^d X[n] = \Theta(B)\epsilon[n] \tag{13}$$

It has been shown in [23] that the autocorrelation function associated with ARFIMA(p,d,q) is given by

$$r_{ARFIMA}[k] = c2^{1+d}\sin(\pi d)\frac{\Gamma(k+d)}{\Gamma(k+1-d)}\Gamma(1-2d) \quad -1 < d < \frac{1}{2}, d \neq 0 \tag{14}$$

for a large k, we know from Sheppard's approximation, that $\frac{\Gamma(k+a)}{\Gamma(k+b)} \sim k^{a-b}$. Accordingly, Eq. (14) becomes

$$r_{ARFIMA}[k] = c2^{1+d}\sin(\pi d)\Gamma(1-2d).k^{2d-1} \tag{15}$$

$$r_{ARFIMA}[k] = f(d)k^{2d-1} \tag{16}$$

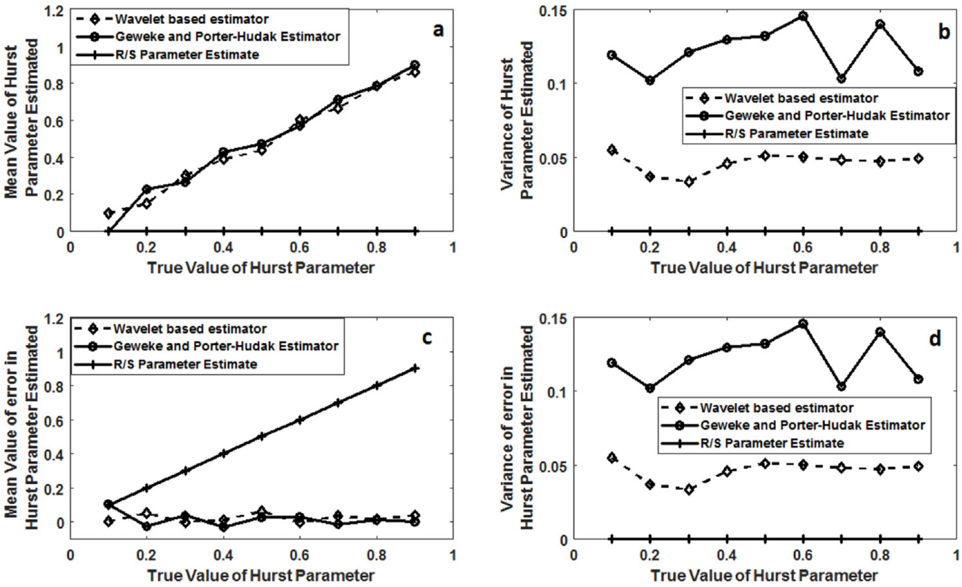

Fig. 3. Results for the estimation of Hürst parameter for the fGn process with length, $L = 64$.

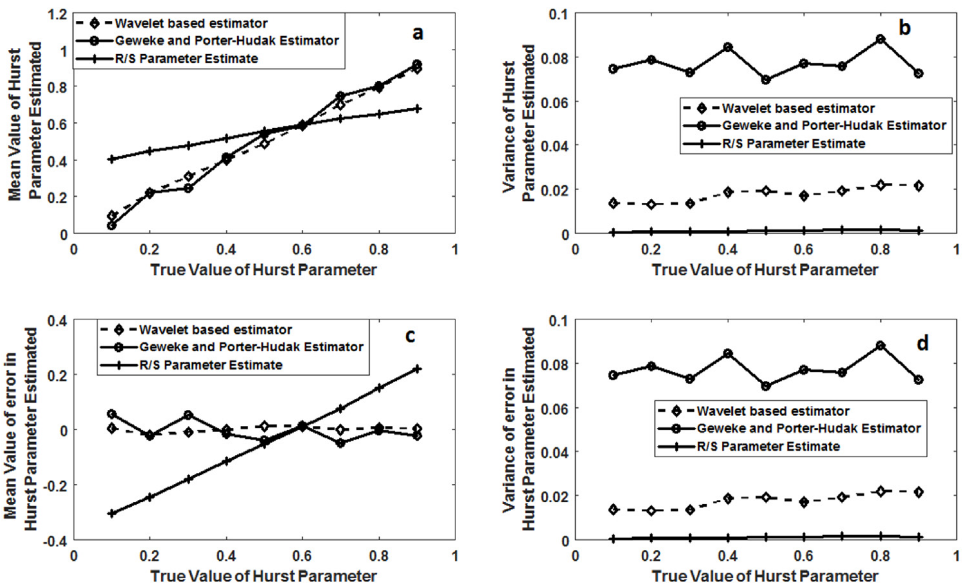

Fig. 4. Results for the estimation of Hürst parameter for the fGn process with length, $L = 128$.

If $0 < d < \frac{1}{2}$, then in Eq. (16), $f(d)$ takes a constant value and causes the autocorrelation function to decay at a rate, which is much slower than the exponential decay rate of ARMA(p,q) processes. So, ARFIMA(p,d,q) models with $0 < d < \frac{1}{2}$ can be used to represent stochastic processes with long-range dependency. It has been shown that if $|d| < \frac{1}{2}$, then ARFIMA(p,d,q) models represent self-similar stationary stochastic processes. The degree of differencing, d and the Hürst parameter, H are related by $d = H - \frac{1}{2}$ [27,31]. Fig. 2 shows the sample path of ARFIMA(0,0.4,0) (Fig. 2a, 2c), ARFIMA(0,−0.4,0) (Fig. 2b, 2d) processes and their corresponding autocorrelation functions. ARFIMA(0,d,0) processes are known as fractionally integrated noise. It can be seen that the autocorrelation function of ARFIMA (0,0.4,0) process is very much similar to a fGn with Hürst parameter $\frac{1}{2} < H < 1$. On the contrary, the autocorrelation function of ARFIMA(0,d,0) process, with $-\frac{1}{2} < d < 0$, is similar to fGn with Hürst parameter $0 < H < \frac{1}{2}$ and is therefore characterized by fast decaying autocorrelation function. In this paper, the autocorrelation function of the DWT coefficients of fGn is studied by fitting the ARFIMA(p,d,q) models and then analyzing the autocorrelation function related to these models. Since, fGn is a wide-sense stationary process, the models with $d = 0$ are considered.

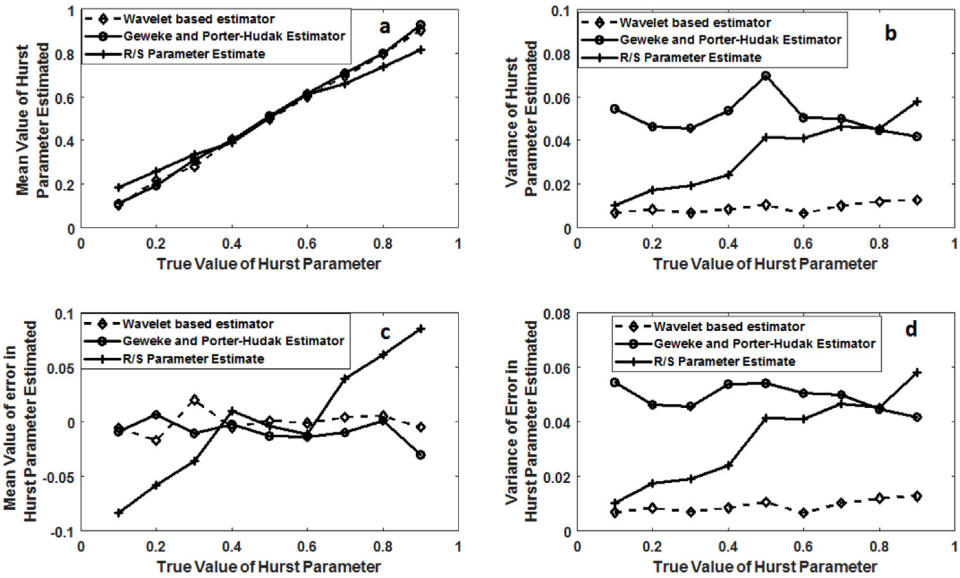

Fig. 5. Results for the estimation of Hürst parameter for the fGn process with length, $L = 256$.

2.2. Wavelet transform of long-memory processes

One can obtain the DWT of fGn using Mallat's recursive algorithm [40]. Let, d_j and a_j denote the detail and approximation coefficients respectively at level j for fGn series, $G_H[n]$. Now, for $j = 0$, $a_0[n] = G_H[n]$. For, $j \geq 1$, the approximation and detail coefficients are obtained recursively using

$$a_{j+1}[n] = \sum_{k=-\infty}^{\infty} h[2n-k]a_j[k] \tag{17}$$

$$d_{j+1}[n] = \sum_{k=-\infty}^{\infty} g[2n-k]a_j[k] \tag{18}$$

where, $g[n]$ and $h[n]$ are quadrature mirror filters related to each other by $g[n] = (-1)^{n-1}h[1-n]$. For the case of fGn, the variance of the detail coefficients has a power law relation with the scale

$$var\{d_j[n]\} = \sigma^2 2^{(2H+1)j} \tag{19}$$

Kaplan et al. [45] showed that, for any wavelet with M vanishing moments, the intra-scale autocorrelation decays with a power law relation i.e. $\mathbb{E}[d_j[k]d_j[l]] \sim \mathcal{O}(|2^j(k-l)|^{2H-M})$. They also demonstrated that if wavelet bases with vanishing moments $M \geq 2$ are selected, then the coefficients are uncorrelated self-similar processes. So, the exact expression for the autocorrelation function corresponding to DWT coefficients of fGn is unknown in the literature. In this paper, the nature of autocorrelation function for intra-scale wavelet coefficient is studied by estimating the parameters of ARFIMA(p,d,q) model fitted to a sample wavelet coefficient of fGn at a level $j \geq 1$. Once the parameters of the models are estimated, one can have an estimate of the nature of autocorrelation function using Eq. (16)

2.3. Methodology

The aim of this paper is to study the nature of autocorrelation function of DWT coefficients of fGn with varying values of Hürst parameter(H) and with different wavelet functions having different vanishing moments. This is accomplished by studying various statistical models like ARMA(p,q) and ARFIMA(p,d,q) that fit the DWT coefficients at a particular level of decomposition using a wavelet with varying orders of vanishing moments.

One hundred sample realizations of fGn with varying values of Hürst parameter were generated using circular embedding [49] for every realization. The differencing parameter was estimated by first finding the estimate of Hürst parameter. Hürst parameter was estimated using different techniques like re-scaled range (R/S) analysis and detrended fluctuation analysis (DFA) in time domain [3,33,34], in frequency domain [35,36] and in wavelet domain [37–39]. The differencing parameter, d, is then estimated using the relation $d = H - \frac{1}{2}$. Figs. 3–8 show results of Hürst parameter estimation for fGn of varying Hürst exponent and varying length. It can be seen that the results are not good, when the length of the sample path, L, is small. Therefore, length of data equal to 2^{10} was considered in further analysis.

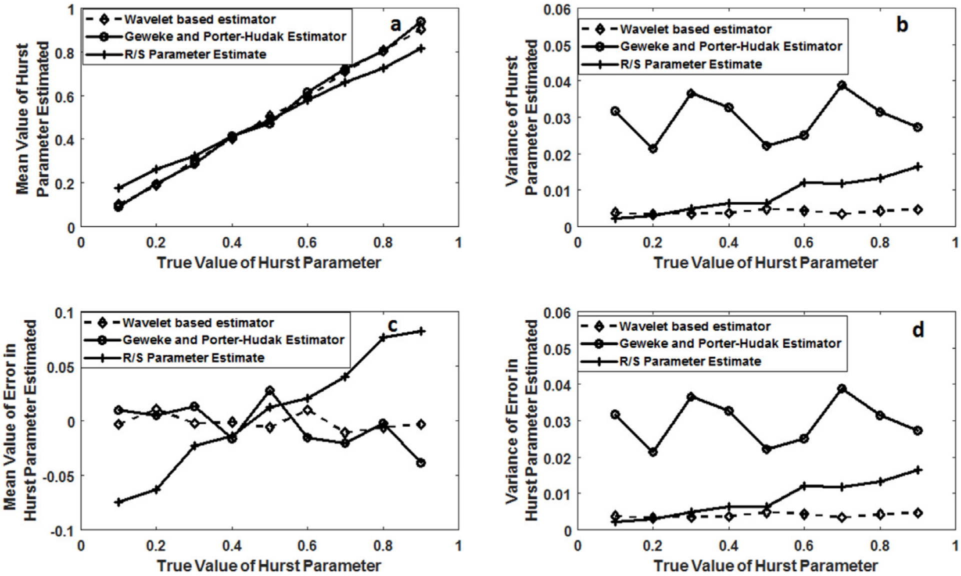

Fig. 6. Results for the estimation of Hürst parameter for the fGn process with length, $L = 512$.

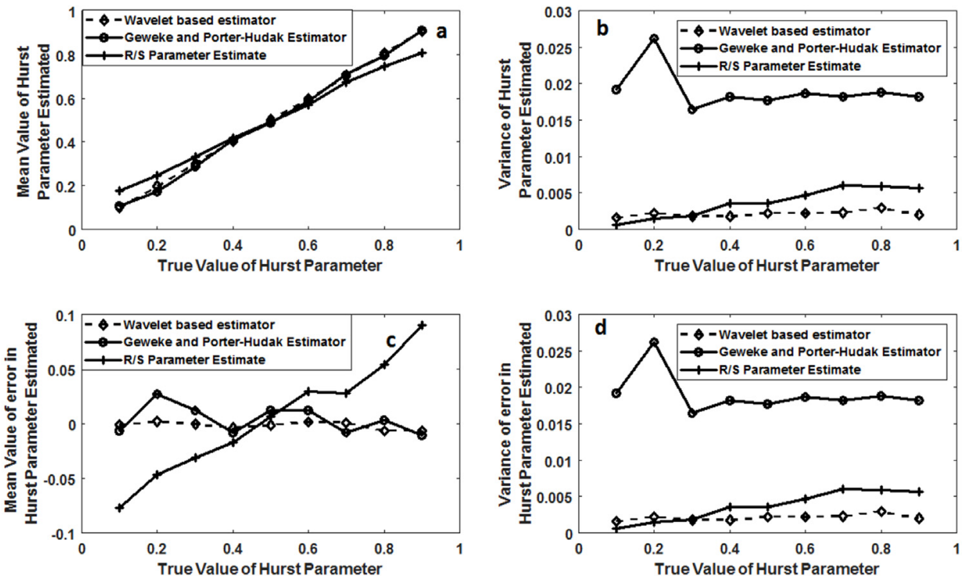

Fig. 7. Results for the estimation of Hürst parameter for the fGn process with length, $L = 1024$. Hurst parameter values obtained in this case is used for developing ARFIMA(p,d,q) models in the manuscript.

For all the realizations corresponding to fGn with a specified Hürst parameter value, a multilevel DWT was computed. For example, corresponding to $H = 0.4$, hundred independent fGn realizations were generated and for each realization, a multilevel (4 levels) DWT was performed. So we have hundred series of DWT coefficients for each level of decomposition. At a level l, the degree of fractional differencing is estimated for each series of DWT coefficients using the wavelet-based estimator. Mean value of the differencing parameter, denoted as $\hat{d}_l$, is then calculated for level l (see Figs. 3–8). Parameters corresponding to $L = 1024$ (Fig. 7) are used for further analysis in this manuscript.

3. Results and discussion

The wavelet transform of self-similar (or a long-memory process) is also wide-sense stationary and self-similar. However, as mentioned above, the exact expression for the autocorrelation function of DWT coefficient corresponding to

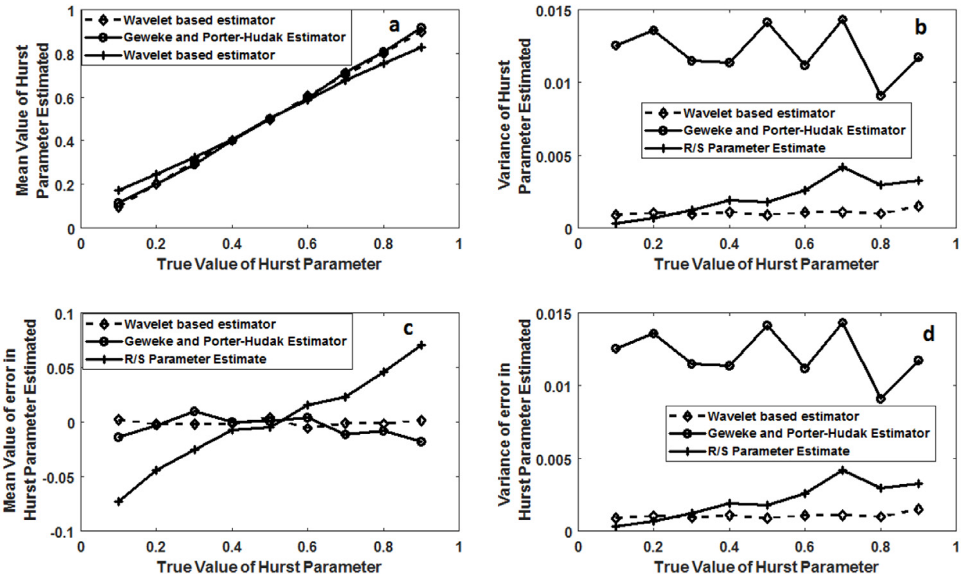

Fig. 8. Results for the estimation of Hürst parameter for the fGn process with length, $L = 2048$.

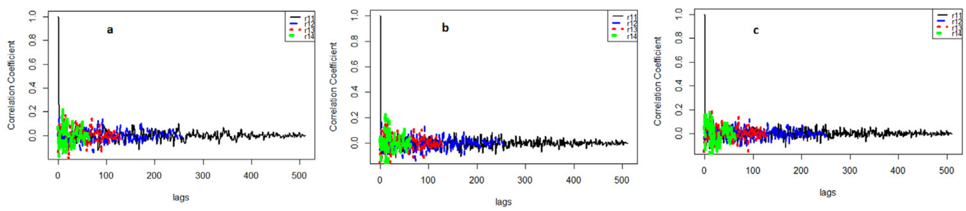

Fig. 9. Inter-scale (r_{11}) and intra-scale cross correlations(r_{12}, r_{13}, r_{14}) of DWT coefficients for fGn with (a) H = 0.1 (b) H = 0.5 (c) H = 0.9 using db-6 wavelet. Here, $r_{1k}[l] = \sum_{k=-\infty}^{\infty} G_{H_{db6_1}}[n]G_{H_{db6_k}}[n-l]$, $k \in 2, 3, 4$. Where $G_{H_{db6_k}}[n]$ denotes the wavelet coefficient of fGn model at a level k.

fGn is not known, except in the case of Haar bases. This is because, for a given signal, the associated Haar approximation and detail coefficients (see (17), (18)) take a very simple analytical form given by :

$$a_{j+1} = \frac{1}{\sqrt{2}}(a_j[2k] + a_j[2k-1]) \tag{20}$$

$$d_{j+1} = \frac{1}{\sqrt{2}}(a_j[2k] - a_j[2k-1]) \tag{21}$$

For wavelets other than Haar (like Daubechies family), such an easy expression cannot be obtained and hence it is hard to generalize the behavior of the inter-scale and intra-scale correlations for wavelets with higher order vanishing moments. This correlation analysis was performed by analyzing different linear stationary models like ARMA(p,q) and ARFIMA(p,d,q) models, where the degree of differencing $|d| \leq \frac{1}{2}$. The ability of these models to fit the DWT coefficients of fGn process was analyzed by observing the values of AIC and BIC associated with each models. The degree of differencing d is estimated using the exact maximum likelihood [31].

Results of estimation of differencing parameter for DWT coefficients of fGn with Hürst parameter varying from 0.1 to 0.9 show that, for a larger degree of differencing in time domain, the degree of differencing of DWT coefficients is more negative. For example, for fGn with $H = 0.1$ the $\hat{d}_l$ estimated for $l = 1, 2, 3, 4$, for wavelets with different vanishing moments (from Haar to db-10) varies from (0.1478–0.3157) with a mean value of 0.1906 and the variance of 1.979×10^{-3}). In the case of fGn with $H = 0.5$, the estimated $\hat{d}_l$ varies from (-5.828×10^{-2}–7.147×10^{-2}) with a mean of 4.271×10^{-4} and variance of 6.546×10^{-4}. Finally for fGn with $H = 0.9$, the estimated $\hat{d}_l$ varies from (-0.1867–0.1030) with a mean of -0.1221 and variance of 6.513×10^{-4}. These values for fGn with $H = 0.2, 0.3, 0.4, 0.6, 0.7, 0.8$ also indicates clearly that the degree of differencing estimated for all the wavelet bases with varying vanishing moments take on negative values with increasing magnitude as the value of the Hürst parameter increases. But the magnitude of the degree of differencing is always follows, $|d| < \frac{1}{2}$. This is in agreement with the results of Beran et al. [27], Kaplan and Kuo [45] and indicates that the DWT coefficients of a long-memory process (process for which $0.5 < H < 1$ or $0 < d < \frac{1}{2}$) are uncorrelated self-similar processes and the larger the value of Hürst parameter, the more decorrelated the DWT coefficients are. On the other hand, the DWT coefficients corresponding to an anti-persistent self-similar process (processes for which $0.1 < H < 0.5$ or $-\frac{1}{2} < d < 0$) have a degree of differencing, which is indicative of a long-memory characteristic. The lesser the value of Hürst parameter the more is the mean degree of differencing estimated at all levels of wavelet decomposition. But the magnitude of degree of differencing estimated for the DWT coefficients in this case always follows $0 < d < \frac{1}{2}$. So, the DWT coefficients for fGn having anti-persistent behavior are also stationary and self-similar but, these coefficients have a long-memory characteristics.

The second step in the analysis was to estimate the parameters of the statistical model that best described the DWT coefficients. It is evident from the values of AIC and BIC for all the sequences of DWT coefficients that the ARFIMA(p,d,q) models with $|d| < \frac{1}{2}$ are best suited to describe the DWT coefficients resulting from the use of wavelet bases with varying number of vanishing moments, for all the fGn processes considered. Decay of intra-scale correlation (or autocorrelation) is slower, compared to the inter-scale correlations. For example, if $G_{H_{db6_1}}$, denotes the DWT coefficients at level-1 decomposition of fGn with H = 0.1 with respect to db-6 wavelet, then the model estimated according to the best AIC for $G_{H_{db6_1}}$, $G_{H_{db6_2}}$, $G_{H_{db6_3}}$ and $G_{H_{db6_4}}$ is given by:

$$
\begin{aligned}
G_{H_{db6_1}} &:= (1 - 1.1009B - 0.5934B^2 + 0.710064B^3) \\
&\quad (1-B)^{0.151453}X[n] = (1 - 1.07216B - 0.7813B^2 + 0.8557B^3)\epsilon[n] \\
&\quad \epsilon[n] \overset{iid}{\sim} \mathcal{N}(0, 0.013) \\
G_{H_{db6_2}} &:= (1-B)^{0.2319}X[n] = (1 + 0.0018B + 0.178161B^2)\epsilon[n] \\
&\quad \epsilon[n] \overset{iid}{\sim} \mathcal{N}(0, 0.0064) \\
G_{H_{db6_3}} &:= (1 + 0.0493B + 0.0701B^2 - 0.8816B^3) \\
&\quad (1-B)^{0.2219}X[n] = (1 - 0.0404B - 0.0798B^2 + 0.8790B^3)\epsilon[n] \\
&\quad \epsilon[n] \overset{iid}{\sim} \mathcal{N}(0, 0.0023) \\
G_{H_{db6_4}} &:= (1 - 0.645B)(1-B)^{0.2602}X[n] = (1 - 0.624B - 0.374B^2)\epsilon[n] \\
&\quad \epsilon[n] \overset{iid}{\sim} \mathcal{N}(0, 0.00133)
\end{aligned}
$$

AIC, BIC pairs for these models were respectively (−2175.25, −2145.15), (−1283.33,-1269.53), (−759.637, −745.851) and (−352.881, −340.65) which was lesser compared to the cases the value of fractional differencing parameter is $d = 0$.

Fig. 9 shows the correlation of level-1 coefficients with level-2 (r_{12}), level-3 (r_{13}) and level-4 (r_{14}) coefficients using db-6 wavelet for DWT analysis for fGn with varying Hürst parameter. In all the cases, the decay of inter-scale correlations is slower than that of the intra-scale correlations. These observations have shown to be consistent with Hsu [50] and Beran et al. [27]. This pattern was observed with all the wavelet bases of Duabechies (db) family.

4. Conclusion

The performance of ARFIMA(p,d,q) models have been tested against ARMA(p,q) (or ARIMA(p,0,q)) and it has been shown that ARFIMA(p,d,q) models with $|d| < \frac{1}{2}$ are selected as optimal models according to the AIC and BIC for modeling the statistics of DWT coefficients of fGn, indicating the efficacy of these models to represent the statistical nature of DWT coefficients of fGn. This is because, the AIC and BIC take less values for ARFIMA(p,d,q) models with $|d| < \frac{1}{2}$ compared to ARMA(p,q) (or ARIMA(p,0,q)) models. Also, these DWT coefficients are uncorrelated self-similar processes with $-\frac{1}{2} < d < 0$, for fGn with the Hürst parameter lying in the range $0.5 < H < 1$. On the other hand, the DWT coefficients of the anti-persistent processes (processes for which $0 < H < 0.5$) are also uncorrelated self-similar processes but these coefficients show a long-memory behavior, as the mean degree of differencing estimated is in the range $0 < d < \frac{1}{2}$. The faster decay of the inter-scale correlation of the DWT coefficients compared to intra-scale correlation corresponding to fGn with varying Hürst parameter is also verified using different models estimated.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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