Application of Fuzzy graph theory in brittle plane network analysis - A potential method for carbon sequestration models

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ABSTRACT

Improved carbon sequestration (CCS) models with rocks as sinks require incorporation of uncertainty into the models. In such cases of uncertain geoscientific problems, fuzzy graph theory can be useful. Brittle shear plane network with indistinct shear planes is common in natural sheared rocks, and can be targeted for CCS. Due to non-unique possibility of continuity of P-planes, it is not possible to represent such networks as crisp graphs. We present few natural examples of the former type of P-planes in shear zones, and how fuzzy graph theory can represent the fracture network and fluid flow. The process involves assigning some sample numerical probability to represent the connectedness between the underdeveloped P-planes and the Y-planes. The presentation is a geometric exercise and does not extend to the genesis of the shear zones.

Keywords: Network, Fluid flow, Structural geology, Shear planes, Geoscientific uncertainty, Waste disposal, Carbon sequestration models

INTRODUCTION

Structural geological modeling requires representation of structures in some numerical form. When the structures are clearly decipherable, this is rather easy, for example, one can apply the graph theory (e.g., Sanderson et al., 2018; Mukherjee, 2019). Classical graph theory works when there is a full certainty that which nodes are connected by which vertices through edges.

Various brittle plane geometries develop in rocks at shallow crustal depths that undergo brittle shear (Figure 1a). In case the shear planes are clearly delineated (Figure 1b), one can use graph theory to represent the brittle plane network in terms of sigmoid shear bound by parallel planar shear planes commonly seen in sheared rocks. For example, in case of Figure 1b, the adjacency matrix is:

	a	b	c	d	e	f
a	0	0	0	0	1	0
b	0	0	0	0	0	1
c	0	0	0	0	0	1
d	0	0	0	0	0	1
e	0	0	0	0	0	1
f	0	0	1	1	0	0

Carbon sequestration (CCS) has recently been modeled in terms of extraneous CO_2 sources and CO_2 sinks in rocks and P-graph modeling approach has been undertaken (Chong et al., 2014). Fractured rocks especially coal has been targeted for CCS (e.g., Chen et al., 2020). However, since connectivity of fractures is uncertain at depth, establishing models on CCS becomes problematic (Figure 1 of Zhang et al., 2009). Pashin et al. (2008) presented a discrete fracture

network model for CCS in coal. Such models would be possible when existence of the sink locations in the coal is fairly well understood. When the existence of the sinks is uncertain, models will nevertheless be required to be built in decision making. Other uncertainties can be *"unproven nature"* of the carbon sequestration technology, economic cost and life cycles of the CCS technologies and so on (Tan et al., 2010 and references therein). Researchers have identified CCS as one of the potential areas where graph theory concept needs more elaboration (e.g., Farid et al., 2021).

Anand et al. (2021) discussed various kinds of uncertainties in parameters in structural geological problems. The sigmoid planes can be indistinct due to *(i)* poor snap quality, *(ii)* they are indeed developed less prominently, or *(iii)* sometimes a part of them is below the ground surface hence not seen. For example, in seismic images, brittle planes may not always be clearly decipherable. This can be because of such brittle planes are below the seismic resolution (Misra and Mukherjee, 2018), and yet the modeler needs to make some presentation of the brittle plane network. In such cases, where the classical graph theory cannot represent the brittle plane network where uncertainty exists about continuation of fractures, fuzzy graph theory should be used.

The concept of fuzzy sets was introduced by Zadeh (1965), which later was well applied to solve several research problems that are uncertain in nature. Crisp set is a well-defined collection of distinct objects. If there is any uncertainty in the set boundaries of objects, the concept of fuzzy set is used. In other words, a fuzzy set is used to represent the degree of membership of any qualitative data. The crisp set is not able to work with scientific problems with inherent uncertainty because it consists of just two

truth values: 0 – false (no) and 1-true (yes). These single values of membership degree are unable to manage the uncertainties. Atanassov (1999) proposed the intuitionistic fuzzy set to manage uncertain situations using an extra degree of membership, defined as the *hesitation margin*. Intuitionistic fuzzy set, is as an extension of Zadeh's theory of fuzzy sets. Compared to classical fuzzy set, it is more flexible and efficient to work with uncertainty due to the presence of hesitation margin. Repository presents the fundamentals of the fuzzy graph theory. A fuzzy graph representing fracture network can constraints fluid flow through such a network.

METHODS

Figure 2a-c present a network of brittle shear planes where not all P-planes can be tracked with confidence. Here the curved P-plane AB joins two sub-parallel Y-planes. However, the other P-planes (CD, KZ, EF, GH and IJ) do not join both the C-planes. In some cases, the P-planes are too close-spaced in order to distinguish them confidently in naked eyes. A simplified situation can be thought in this case (Figure 3).
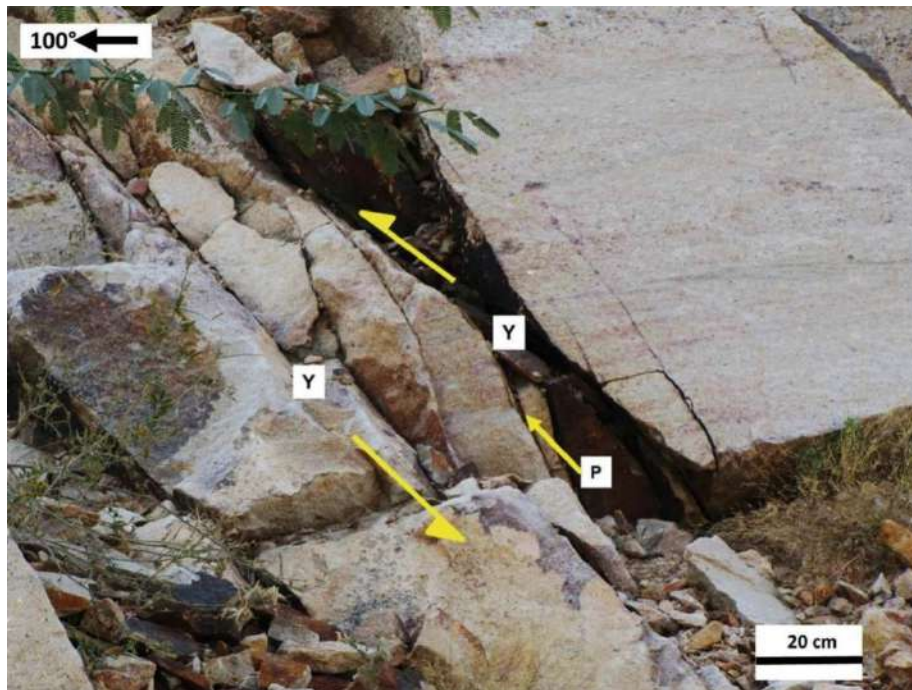

Figure 1(a). Clearly visible Y and P planes. Top-to-ESE brittle shear with distinct Y- and P-planes within fluvial sandstone of the Sarnoo hill area, observed on a sub-vertical section. Location: W of Sarnoo hill area, SE of the Sarnoo village, Barmer basin, India. Reproduced from figure 1.18 of Mukherjee et al. (2020).
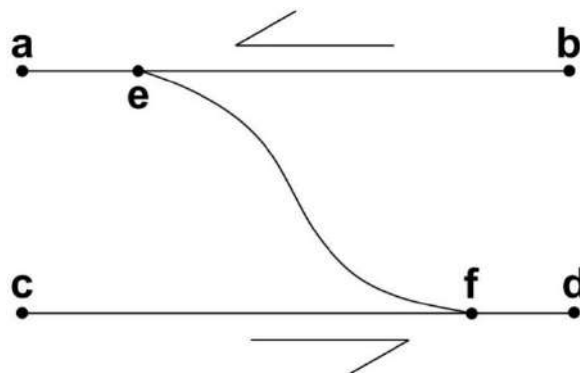

Figure 1(b). A brittle plane network in a brittle shear zone. Reproduced from Figure 2c of Mukherjee (2019).
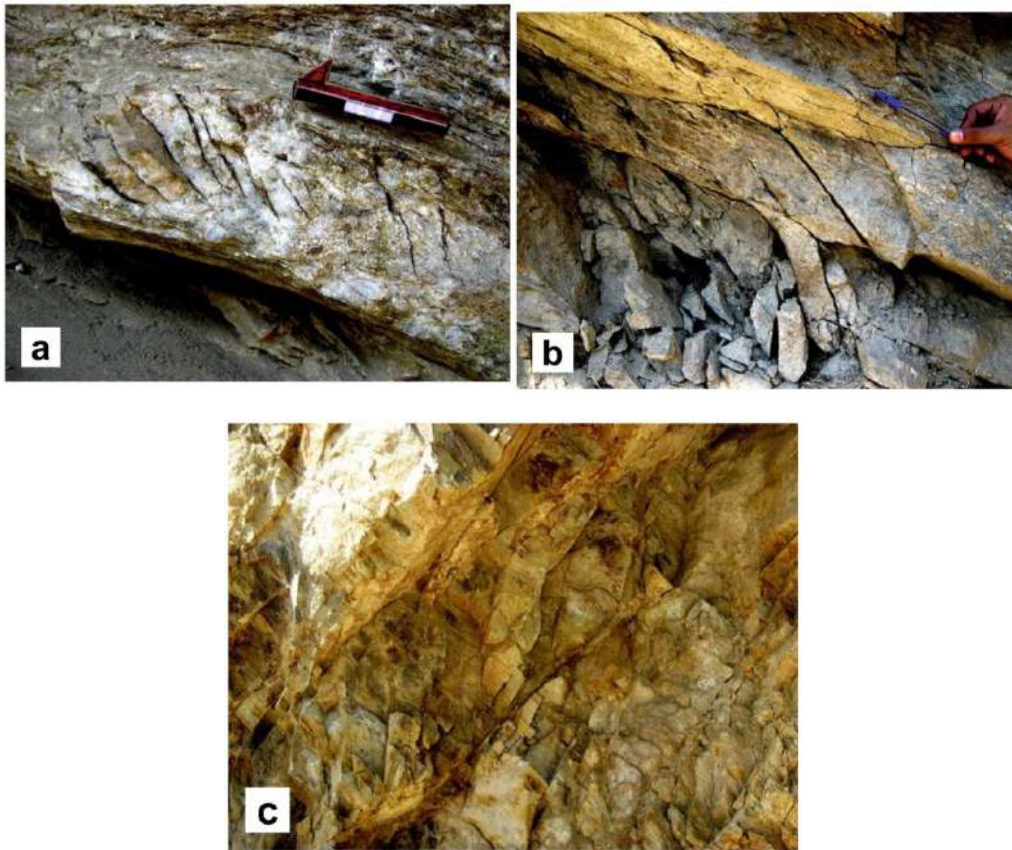
Figure 2. Brittle plane network consisting of Y and P planes. Here some of these planes cannot be traced continuously with full confidence, and hence a crisp graph cannot be visualized. Fuzzy graph theory can be applied in these cases. All examples from mylonitized gneiss in Greater Himalayan Crystallines, Bhagirathi section, Uttarakhand, India- **(a,b)** Top-to-left (up) shear. India; reproduced from figures. 5.15 and 5.20, respectively of Mukherjee (2014). **(c)** Top-to-right (up) shear; reproduced from figure 5.21 of Mukherjee (2014).
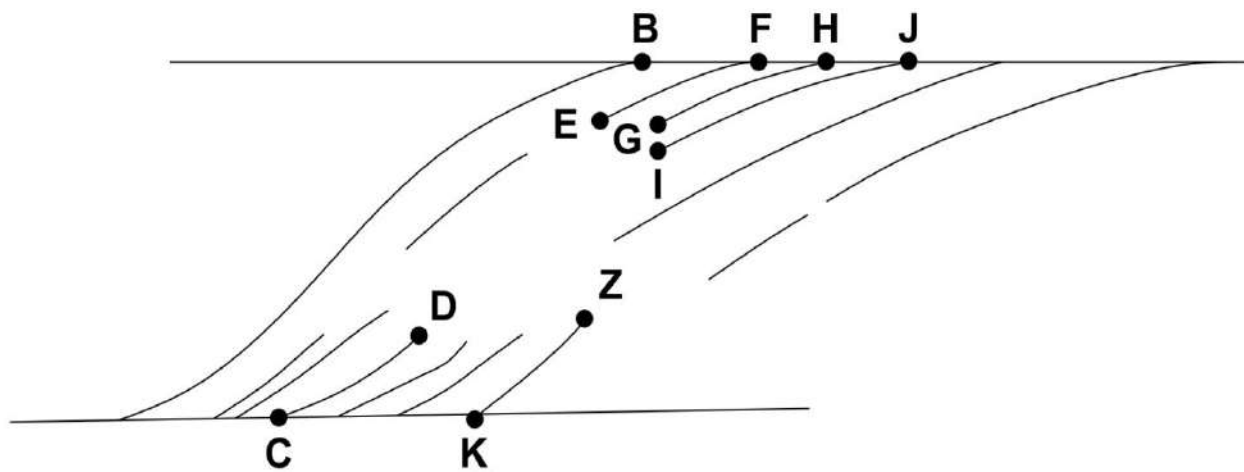

Figure 3. An idealized brittle shear plane network. Few sigmoid P-planes (e.g., AB) join the C-planes. The other P-plane, CD, KZ, EF, GH and IJ, are not fully developed and merge with just a single C-plane. In such a situation, fuzzy graph theory has been applied in this work.

We represent fracture network using intuitionistic fuzzy graph as follows (Figure 4). Points (nodes) A and B are certainly connected with AB as an arc. But it is uncertain for some reason whether point D connects with E, G or I by some arc. Connection between D and E or I, therefore can be represented as an intuitionistic fuzzy graph. Now if by zooming the image or by some image enhancement technique, a connection between D and G is established with some degree of confidence, it means that there is a much higher probability/possibility that they are connected, and therefore connections between D and E or D and I would be less likely/less probable. Putting some sample numerical probability values in a scale of 0 ("certainly not") to 1 ("certainly"), or using intuitionistic fuzzy graph enables one to represent the fracture network. In this graph, take the node set as V = {A, B, C, D, E, F, G, H, I, J, K, Z} and A (0.9, 0.1) denotes the degree of membership and degree of non-membership of the node *A* and similarly for all other nodes The ordered pair (0.7, 0.1) denotes the degree of membership and degree of non-membership of the arc relation $e_{FH}=(F,H)$ on $V \times V$ and so on. These numerical values of possibilities are given according to the degree of

confidence of the connection. The values are taken in the interval [0,1].

We will now get into further detail of fuzzy graph theory to tackle flow problems. The adjacency matrix of a fuzzy graph $G :(V,\sigma,\mu)$ is an $n \times n$ matrix defined as $A=[a_{ij}]$,

where $a_{ij}=\mu\ (v_i,v_j)$ (Anjali and Mathew, 2013). For example, consider the fuzzy graph in Figure 5 for a hypothetical data set. Its adjacency matrix is:

$$
\begin{array}{c c c c c}
 & v_1 & v_2 & v_3 & v_4 \\
v_1 & 0 & 0.1 & 0.9 & 0.4 \\
v_2 & 0.1 & 0 & 0.6 & 0 \\
v_3 & 0.9 & 0.6 & 0 & 0.2 \\
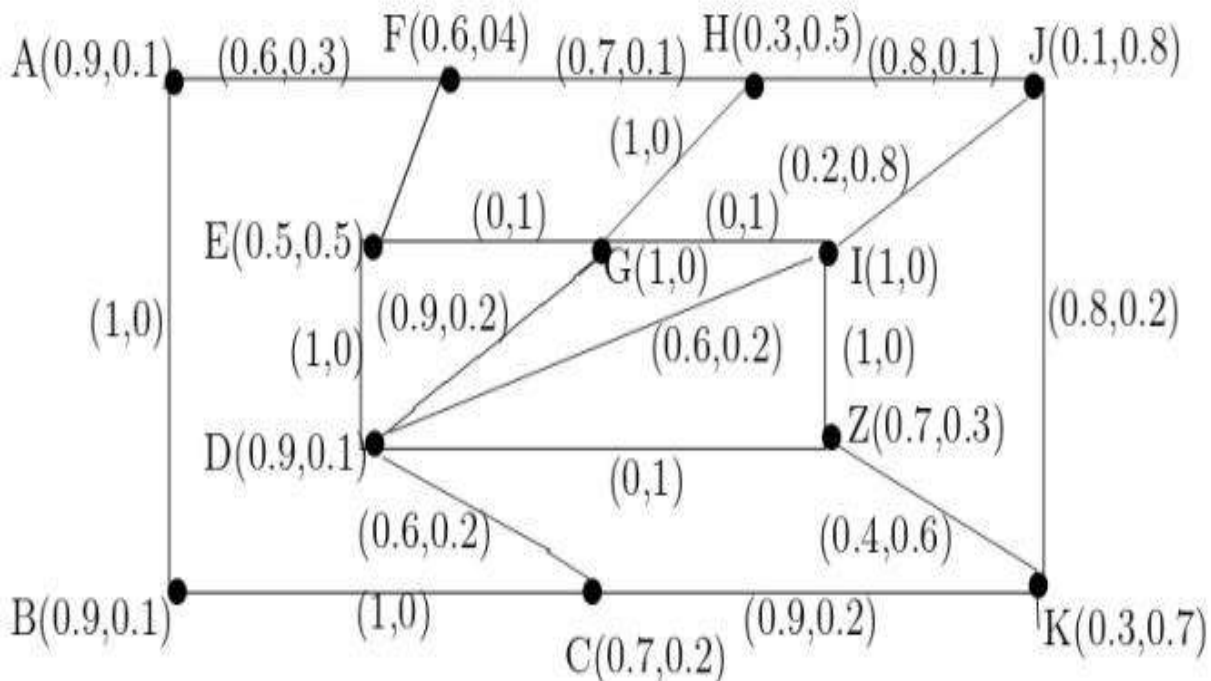v_4 & 0.4 & 0 & 0.2 & 0
\end{array}
$$

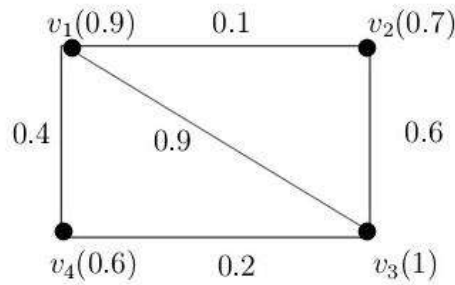**Figure 4.** A fuzzy graph corresponding to the case of Figure 3.

Figure 5. A fuzzy graph with nodes v_1, v_2, v_3 and v_4. Values inside brackets of the nodes are the fuzzy weights of the respective nodes. Fuzzy arc weights are represented by the values 0.1, 0.2, 0.4, 0.6, and 0.9.
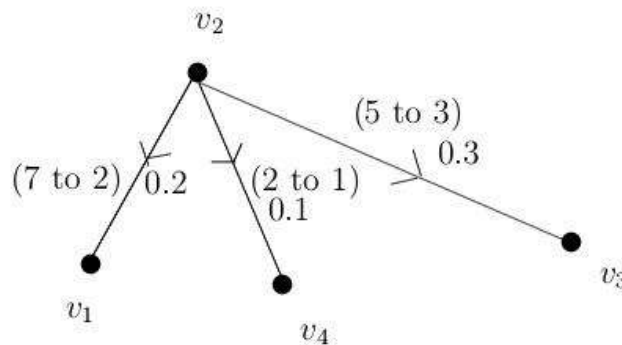

Figure 6. A network, from nodes v_2 to v_1 the flow can vary in magnitude from 7 to 2 units. Therefore, we take fuzzy weights 0.7 for v_2 and 0.2 for v_1. Similarly, one can take the fuzzy weight of v_3 as 0.3, v_4 as 0.1.

In this fuzzy graph, the numbers inside the first bracket represents the node weights, which do not appear in the adjacency matrix since. Instead, only the arc weights are used for the entries in the adjacency matrix. Such a presentation is as per Anjali and Mathew (2013). The entries in the adjacency matrix are obtained as follows. Since v_1 is not shown to be connected with itself by any edge, the entry is 0. Since v_1 is adjacent with v_2 with arc weight 0.1, the entry is 0.1 and so on.

Consider the network in Figure 6. From v_2 to v_1, the flow can vary in magnitude from 7 to 2 units. Therefore, we take fuzzy weights 0.7 for v_2 and 0.2 for v_1. Similarly, one can take the fuzzy weight of v_3 as 0.3, v_4 as 0.1. This is a first-time approach through this work. Suppose the fuzzy graph presented in Figure 6 has no arrows on edges. Then its corresponding adjacency matrix will differ.

That means flow can happen from one vertex to another and vice versa. The definition of a fuzzy graph allows us to give the arc weights in the following way. The arc (edge) weights are taken as the minimum among the corresponding node weights. This is logically true, since the maximum flow between two nodes is the minimum capacity between the two nodes. In that case, the adjacency matrix is:

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0.3 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

Suppose, instead of a range of flow, definite units of flow were given (e.g., 7, 5 and 2). We then assign fuzzy weights as 0.7, 0.5 and 0.2, respectively, for arcs. If a range of flow is given, we will get a freedom to take the minimum value amongst the range for representing the arc weight for the range of flow. As per the definition of adjacency matrix, one can represent these values by using their corresponding fuzzy weights. If 2, 1 and 3 are avoided, i.e., flow range 7 to 2 becomes 7, flow range 2 to 1 becomes 2 and flow range 5

to 3 becomes 5, the flow becomes exact, and one can assign the fuzzy weights. In fuzzy graph, we can give the maximum value of node weight and arc weight as 1 and minimum as 0. All other values are in between 1 and 0. A value equal to 1 means that the flow has fully happened. The intensity of the flow is represented by using these fuzzy weights.

Ramakrishnan and Lakshmi (2008) discussed how to fuse two nodes of the same fuzzy graph as follows. Let $G: (V, \sigma, \mu)$ be a fuzzy graph and let $u, v \in \sigma$. By the join (fusion) of two vertices u and v, the following is meant.

(i) Fuse (join) the vertices u and v as uv in the corresponding crisp graph $G^* = (\sigma^*, \mu^*)$ and then consider its underlying simple graph. A simple graph is the one without loops and possessing multiple edges.

(ii) The resulting fuzzy graph is $G_{uv}=(\sigma_{uv}, \mu_{uv})$ where
$\sigma_{uv}(x) = \{\max[\sigma(u), \sigma(v)]$ if $x = uv$

$\sigma(x)$ if $x \neq uv\}$

and $\mu_{uv}(x, y) = \{\max[\mu(x, u), \mu(x, v)]$ if $y = uv$

$\max[\mu(u, y), \mu(v, y)]$ if $x = uv$

$\mu(x, y)$ if $x \neq uv$, $y \neq uv\}$

Consider the fuzzy graph in Figure 7. Figure 8 represents the fusing of the vertices a and b.

For instance,

$\mu_{ab}(ab,d)=Max\{\mu(a,d),\mu(b,d)\}=Max\{0.5,0\}=0.5$

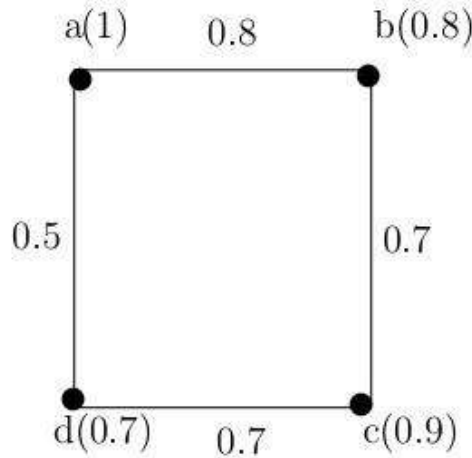$\sigma_{ab}(ab)=Max\{\sigma(a),\sigma(b)\}=Max\{1,0.8\}=1$

Figure 7. Example of a fuzzy graph with nodes a, b, c and d. Values inside the brackets of the nodes represent the fuzzy weights. Fuzzy arc weights are 0.5, 0.7 and 0.8.
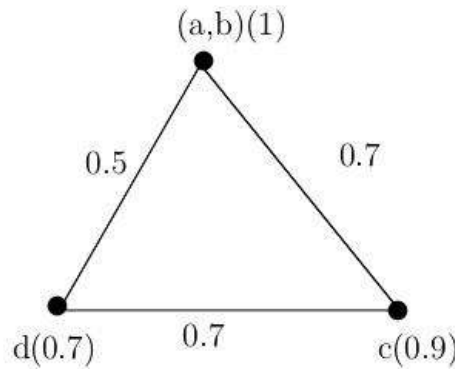

Figure 8. Fusing nodes a and b in the graph presented in Figure 7.

By using this idea, we can define the fusion of two vertices from two fuzzy graphs (Figure 9). Consider two fuzzy graphs G_1: (V_1, σ_1, μ_1) and G_2: (V_2, σ_2, μ_2). Let $u \in G_1$, $v \in G_2$. Fuse (join) the vertices u and v as uv in the corresponding crisp graph and then consider the underlying crisp simple graph. The resulting fuzzy graph is $G_{uv}=(\sigma_{uv}, \mu_{uv})$ where:

$\sigma_{uv}(x) = \{\max[\sigma_1(u), \sigma_2(v)]$ if $x = uv$

$\sigma_1(x)$ if $x \in G_1$, $x \neq uv$

$\sigma_2(x)$ if $x \in G_2$, $x \neq uv\}$

$\mu_{uv}(x,y) = \{\mu_1(x,u)$ if $y = uv$, $x \in G_1$

$\mu_2(x,v)$ if $y = uv$, $x \in G_2$

$\mu_1(x,y)$ if $x,y \in G_1$

$\mu_2(x,y)$ if $x,y \in G_2$ $\}$

Coming to the case of a directed fuzzy graph or a fuzzy digraph, consider the network in Figure 10. The adjacency matrix of this new fuzzy graph is given below.

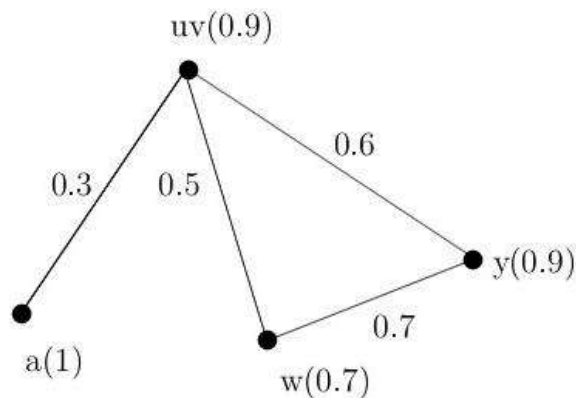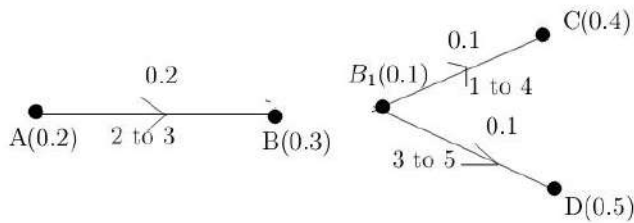|	A	BB_1	C	D
A	0	0.2	0	0
BB_1	0	0	0.1	0.1
C	0	0	0	0
D	0	0	0	0
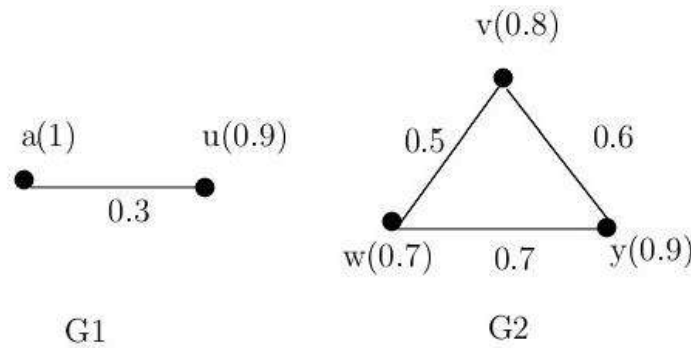
Figure 9. Fusion of two vertices u and v from two fuzzy graphs G_1 and G_2, respectively into uv. The fused vertex is assigned a weight 0.9, which is the maximum weight in between u and v.
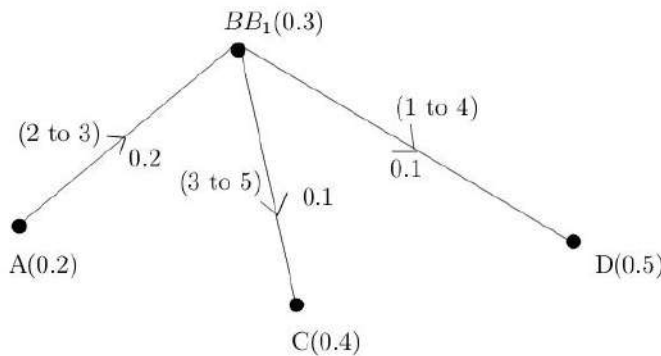
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Figure 10. Fusing two nodes B and B₁ from two different networks. In the BB1 fused vertex, 0.3 is stated inside bracket. The magnitude is the maximum out of B (0.3) and B1 (0.1).

In the BB1 fused vertex, 0.3 is stated inside bracket. This means that the maximum out of B (0.3) and B1 (0.1) to be stated: "BB1 (0.3)". For the maximum taken, the logic is that the capacity elevates when nodes are fused (Ramakrishnan and Lakshmi, 2008).

The general expression for fusing n number of vertices is defined as follows. Fusing of n number of nodes within the same fuzzy graph: Let $G = (V,\sigma,\mu)$ be a fuzzy graph and let $u_1, u_2, \ldots . u_n \in \sigma^*$. By the join of n vertices $u_1, u_2 \ldots, u_n$ we mean:

(i) Fuse the vertices u_1, u_2, u_n in the corresponding crisp graph $G^* = (\sigma^*, \mu^*)$

(ii) The resulting fuzzy graph is

$Gu_1u_2\ldots u_n=(\sigma u_1 u_2 \ldots \ldots \ldots u_n, \mu u_1 u_2 \ldots \ldots u_n)$

Here, $\sigma_{u_1 u_2 \ldots u_n}(x) = \{\max[\sigma(u_1), \sigma(u_2), \ldots \ldots \sigma(u_n)]$

if $x = u_1 u_2 \ldots \ldots u_n$

$\sigma(x)$ if $x \neq u_1 u_2 \ldots \ldots u_n\}$ and

$\mu_{u_1 u_2 \ldots \ldots u_n}(x,y) = \{\max[\mu(x, u_1), \mu(x, u_2),$
$\ldots \ldots \mu(x, u_n)]$

if $y = u_1 u_2 \ldots \ldots u_n$

$\max[\mu(u_1, y), \mu(u_2, y) \ldots \ldots \mu(u_n, y)]$

if $x = u_1 u_2 \ldots \ldots u_n$

$\mu(x, y)$

if $x \neq u_1 u_2 \ldots \ldots u_n$, $y \neq u_1 u_2 \ldots \ldots u_n\}$

SUMMARY

Indistinct or poorly developed brittle plane networks are common in rocks. Their numerical representation can be important for modelers, such as in fluid flow, CCS and hydrocarbon reservoir studies. Hydrocarbon resources are depleting, and flow models realistic to geologic cases are of paramount importance. In this article we introduce the concept of fuzzy graph theory as a first step to fulfill such a far-reaching aim. An example of brittle shear planes consisting of Y and P-planes was considered with different degrees of connection between them. We further presented theoretical issues regarding fuzzy digraphs and joining / fusion of vertices.

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Compliance with Ethical Standards

The authors declare no conflict of interest and adhere to copyright norms.

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