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SHORT NOTE

Locating center of gravity in geological contexts

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Abstract

This short article at first deduces center of gravity (C.G.) for horizontal single-layered and multi-layered rock bodies. Spatial variation of density is also incorporated using geologically realistic density–depth linear and exponential algebraic relations. These are the two well-known depth-wise density variation patterns. The derivations are then extended for dipping rock bodies. Location of C.G. depends on the (i) dimension (length, width and thickness) and (ii) dip amount of the geological body, (iii) the densities of the solid matrix and the pore fluid of one or more than one layers, (iv) gradient of density change with depth and along other two perpendicular directions-for individual layers, and for sedimentary units (v) on the porosity at the surface and (vi) the compaction constant. This work would be useful in gravity- and isostatic studies, and in deciphering stabilities of crustal blocks.

Keywords Centroid · Statics · Density gradient · Pore fluid density · Matrix density · Stability

Introduction

Center of gravity (C.G.) is a conceptual point in a body at which its weight can be supposed to act (Das and Mukherjee 2012). Understanding location of the C.G. of rock bodies and structures (such as a folded rock body) is of significant importance, mainly in tectonics, geodesy and engineering geology to estimate/analyze structural stabilities (e.g., Hwang and Hsiao 2003; Denny 2010; de Blasio 2011; Bock and Melgar 2016). In some cases, the body/structure overturns if the C.G. lies outside it (Hurol 2016). If the gravitational spreading process operates on a geological body for a very long geological time, such as thousands of year or even more, its C.G. descends (Peel 2014). Deducing C.G. of solids with uniform and non-uniform masses to understand their stability is also a standard exercise in statics (e.g., Das and Mukherjee 2012) and geodesy. Second, the pendulum bob is attracted by the C.G. of geological bodies, since the mass of the body concentrates (theoretically) at that point. To substantiate gravity and isostatic studies (Holmes 1945), therefore, knowing the location of the C.G. becomes important.

This short article deduces C.G. for layered horizontal rock types (case-i), rock later exponentially (case-iia) and linearly (case-iib) varying density downward. In addition, relocation of C.G. due to tilting of rock layers (case-iii) is also considered.

Derivations

Case-i Layered lithology, each with constant density, say rock layers 1 and 2 have length 'x_1', width 'y_1', and depths 'q' and '$z_1 - q$', respectively (Fig. 1). ρ_1, ρ_2 are densities of respective layers. 'g' is acceleration due to gravity. The total thickness of the two layers is 'z_1'. Mass of blocks 1 and 2:

$$M_1 = x_1 y_1 q g \rho_1, \tag{1}$$

$$M_2 = x_1 y_1 (z_1 - q) g \rho_2. \tag{2}$$

Coordinate of C.G. of layer-1 becomes

G_1 : $[0.5x_1, 0.5y_1, 0.5q]$ (expression 1)

And that of layer-2

G_2 : $[0.5x_1, 0.5y_1, 0.5(z_1 + q)]$ (expression 2)

The distance between these two C.G.s is '$0.5z_1$'. Say G is the resultant C.G. of the two layers taken together. Then,

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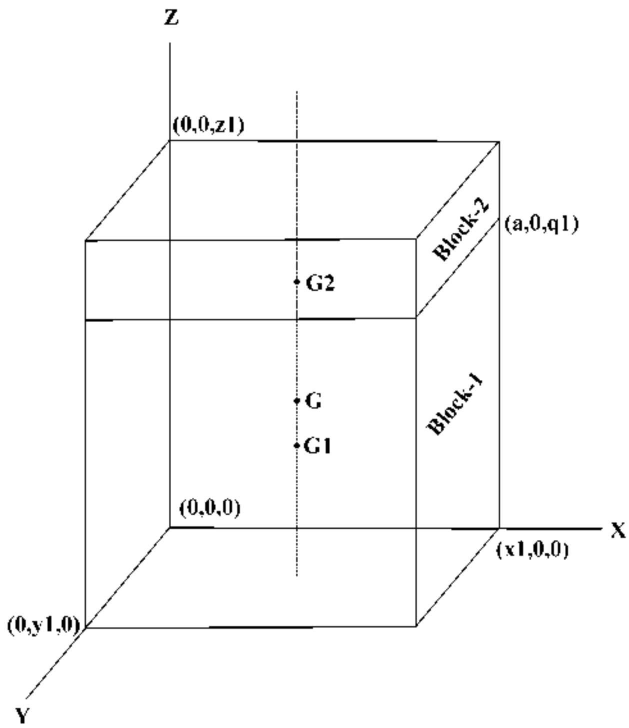

Fig. 1 Two-layered horizontal rock body. Density of each layers are constant. Vide "*Case-i*" in main text for detail

$$G_1G + G_2G = 0.5z_1. \tag{3}$$

Now,

$$M_1M_2^{-1} = G_2G/G_1G. \tag{4}$$

Or,

$$(M_1 + M_2)M_2^{-1} = (G_2G + G_1G)\,G_1G^{-1}. \tag{5}$$

Using Eq. 3, writing q as q_1, and $(z_1 - q) = q_2$,

$$G_1G = 0.5q_2\rho_2(q_1 + q_2)\{q_1\rho_1 + q_2\rho_2\}^{-1}. \tag{6}$$

Adding this with the Z-ordinate of G_1, the Z-ordinate of G is obtained. The full coordinate of G is thus

$$[0.5x_1,\ 0.5y_1,\ \{0.5q_1 + 0.5q_2\rho_2z_1(q_1\rho_1 + q_2\rho_2)^{-1}\}]. \quad \text{(expression 3)}$$

Note (i) for q = 0, i.e., only the layer-1 existing, G coordinate becomes ($0.5x_1$, $0.5y_1$, $0.5z_1$) (expression 4), which validates the derivation. (ii) Depending on magnitudes of q, z_1, ρ_1 and ρ_2, G lies either inside layer-1, or layer-2 or at their interface. Proceeding this way, the C.G. of a rock body with more than two layers can also be found. Note the derivation uses well-known method of finding C.G. of composite bodies as described in static texts.

Case-iia Consider a more general case of density variation in a single rectangular parallelepiped-shaped rock/sediment body (Fig. 2). One of the possible relations among

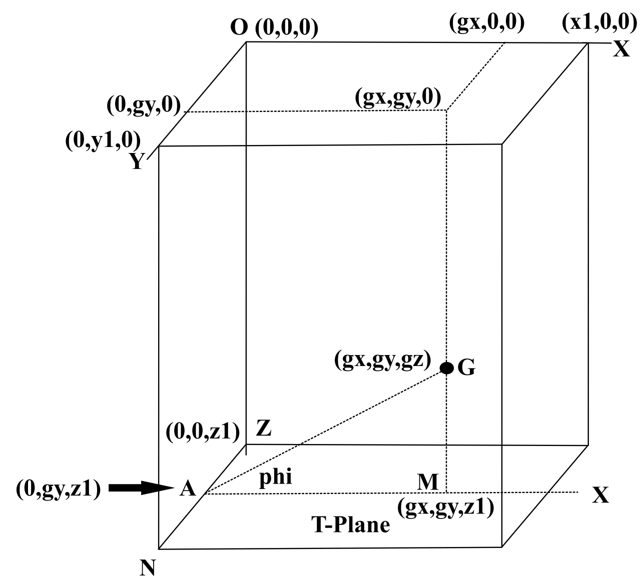

Fig. 2 A single-layered horizontal rock body. Density varies in three perpendicular (*X*, *Y* and *Z*) directions. *G″* Center of gravity. Refer to Cases iia and -b in the main text

wet/fluid-saturated bulk density of sediments/rocks (ρ_{bwz}), matrix/mineral grain density (ρ_m), density of fluid in pores (ρ_f), fractional porosity at surface ($\emptyset_0$) and depth (z) is, as per "Appendix" of this article:

$$\rho_{bw}(0,0,z) = \rho_{bwz} = \rho_m - (\rho_m - \rho_f)\emptyset_0 e^{-bz} \text{ (eqn 4 in Mukherjee 2017a)} \tag{7}$$

Here 'b' is a constant. Considering ρ_m and ρ_f are important since wet density of sediments differ significantly with increasing depth (Jacobi and Smilde 2009). Consider in two perpendicular horizontal directions density varies as follows:

$$\rho_{bw}(x,0,0) = \rho_m - (\rho_m - \rho_f)\emptyset_0 + k_x \cdot x, \tag{8}$$

$$\rho_{bw}(0,y,0) = \rho_m - (\rho_m - \rho_f)\emptyset_0 + k_y \cdot y. \tag{9}$$

Here 'k_i' is the density gradient along 'i'-direction ($i = x$, y). Density at origin (0, 0, 0) is set as

$$\rho_{bw}(0,0,0) = \rho_m - (\rho_m - \rho_f)\emptyset_0. \tag{10}$$

Lateral variation of density along X- and Y-directions can be due to excess pore fluid pressure in sediments (Buryakovsky et al. 1995).

From previous eqns,

$$\rho_{bw}(x,y,z) = \rho_m - (\rho_m - \rho_f)\emptyset_0 e^{-bz} + k_x \cdot x + k_y \cdot y, \tag{11}$$

$$\text{Mass of this parallelepiped is}: M = \int_0^{x_1}\int_0^{y_1}\int_0^{z_1} \rho_{bw}(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{12}$$

Using Eqs. (10) and (11),

$$M = x_1y_1z_1[\rho_m + 0.5(k_x \cdot x_1 + k_y \cdot y_1] + (\rho_m - \rho_f)\emptyset_0 x_1y_1b^{-1}(e^{-bz1} - 1). \tag{13}$$

Now moment about planes *xy, yz* and *zx*, where $r = x, y, z$,

$$M_r = \int_0^{x_1}\int_0^{y_1}\int_0^{z_1} r\rho_{bw}(x, y, z)\,dxdydz. \tag{14}$$

Eqs. (13) and (14) yield

$$M_x = x_1^2y_1[0.5\rho_m z_1 - 0.5b^{-1}(\rho_m - \rho_f)\emptyset_0\left(1 - e^{-bz1}\right) + 0.33k_x x_1 z_1 + 0.25k_y y_1 z_1, \tag{15}$$

$$M_y = x_1{y_1}^2[0.5\rho_m z_1 - 0.5b^{-1}(\rho_m - \rho_f)\emptyset_0\left(1 - e^{-bz1}\right) + 0.25k_x x_1 z_1 + 0.33k_y y_1 z_1, \tag{16}$$

$$M_z = x_1y_1[0.5\rho_m z_1^2 + (\rho_m - \rho_f)\emptyset_0 b^{-1}e^{-bz1}\left(z + b^{-1}\right) + 0.25k_x x_1 z_1^2 + 0.25k_y x_y z_1^2. \tag{17}$$

The coordinate of the C.G. is given by

$[M_r M^{-1}]$. (expression 5)

Case-iib With reference to Fig. 2, say density varies linearly along all the three perpendicular directions, including the vertical one. In that case, Eq. (10) would be replaced by

$$\rho_{bw}(0, 0, z) = \rho_m - (\rho_m - \rho_f)\emptyset_0 + k_z \cdot z. \tag{18}$$

An (empirical) mathematical relation on density variation of rock along some particular direction due to change in mineralogy can also be accommodated likewise Eq. (11) replaces to

$$\rho_{bwz}(x, y, z) = \rho_m - (\rho_m - \rho_f)\emptyset_0 + k_x \cdot x + k_y \cdot y + k_z \cdot z. \tag{19}$$

Tenzer et al. (2012) referred $k_z = 13 \pm 2$ kg m^{-3} km^{-1}, but the magnitude would vary presumably from place to place. Using Eqs. (12) and (18), the mass of the parallelepiped in this case would be

$$M' = x_1y_1z_1\left\{\rho_m - (\rho_m - \rho_f)\emptyset_0 + 0.5(k_x x_1 + k_y y_1 + k_z z_1)\right\}. \tag{20}$$

Using Eqs. (14) and (18),

$$M'_r = {r_1}^2 s_1 m_1\left[0.5\left\{\rho_m - (\rho_m - \rho_f)\emptyset_0\right\} + 0.33k_r r_1 + 0.25\left(k_y s_1 + k_z m_1\right)\right]. \tag{21}$$

When $r = x$, '*s*' and '*m*' are '*y*' and '*z*'. When $r = y$, '*s*' and '*m*' are '*z*' and '*x*'. When $r = z$, '*s*' and '*m*' are '*x*' and '*y*'. As in expression 5, here too the C.G. is given by $[M'_r M'^{-1}]$.

Case-iii Locating the C.G. of a single or a multi-layered horizontal rock body, one can expand the work for dipping lithology as follows. Let's tilt this rock body, without any movement of the margin AN so that it now dips towards left at an angle θ (Fig. 3). Line AN is parallel to the *Y*-axis. Point *G* now shifts to *G′*. The *Y*-ordinate of *G′* remains same as that of the point *G*. However, the orthogonal axes *X, Y* and *Z*, and the reference line AX are not rotated. Say $G\,[g_r]$ is the C.G. of a single or a multi-layered horizontal rock body (Fig. 2) with or without intra-layer variation in density. Point *A* is chosen having a coordinate $(0, g_y, z_1)$. Say line AG makes an angle Φ with the *X*-axis. Point *M* on line AX has a coordinate (g_x, g_y, z_1). In other words, Point *M* is the vertical projection of point *G* on the basal *T*-plane of the parallelepiped. From the coordinates of *A* and *G*,

$$\text{AG distance} = \left[g_x^2 + (z_1 - g_z)^2\right]^{0.5}. \tag{22}$$

Note

$$\text{Cos}\Phi = g_x/\text{AG}, \tag{23}$$

$$\text{Sin}\Phi = (z_1 - g_z)/\text{AG}. \tag{24}$$

$\angle G'AX$ equals $(\theta + \Phi)$. Point M' is the vertical projection of *G′* on the basal plane. *M′* lies on the AM line.

From triangle,

$$\text{AG}'M',\ G'M' = \text{AG}' \cdot \sin(\theta + \Phi). \tag{25}$$

Now since by tilting, a rigid body rotation was induced in the body, AG′ = AG. Therefore,

$$G\prime M\prime = \text{AG} \cdot \sin(\theta + \Phi). \tag{26}$$

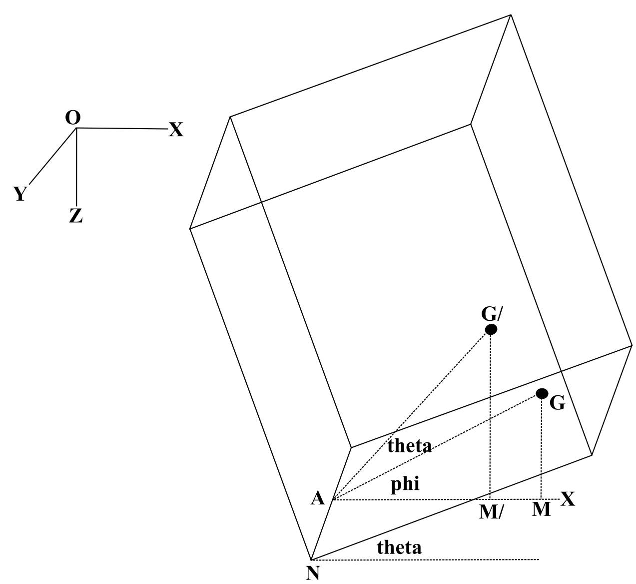

Fig. 3 A rock body, with dip towards left. G' center of gravity

The Z-ordinate of G' is $[z_1 - G/M']$. Using Eqs. (22), (23) and (24), this becomes

$$[z_1 - [g_x \cdot \sin\theta - (z_1 - g_z)\cos\theta]. \quad \text{(expression 6)}$$

Distance AM′ = AG·Cos($\theta + \Phi$) is the X-ordinate of G', which is as follows using Eqs. (23) and (24)

$$[g_x \cdot \cos\theta - (z_1 - g_z)\sin\theta]. \quad \text{(expression 7)}$$

Therefore, the complete co-ordinate of G' is

$$\{g_x \cdot \cos\theta - (z_1 - g_z)\sin\theta\}, g_y, [z_1 - \{g_x \cdot \sin\theta - (z_1 - g_z)\cos\theta]. \quad \text{(expression 8)}$$

To locate G' in geological cases of dipping unit, one needs to consider Y-axis to be parallel to the strike of the dipping plane, X-axis along the dip direction, and Z-axis vertical. Convert the dipping unit to be horizontal first, find out the G coordinate following, for example, cases-i and -ii above, AG length and Φ.

Discussion and conclusions

The centers of gravity (C.G.s) are derived for the following two broad contexts: (i) layered rock with each layer having constant densities; and (ii) a single sedimentary layer with (iia) linearly varying densities along two horizontal directions and exponential increase down the depth; and (iib) linearly varying density in all the three directions. In case (i), C.G. can locate in any of the layers or at their interface. In case (iia), the C.G. does not necessarily coincide with the center of the parallelepiped, which is $[0.5r_1]$. Following statics, this point would be called its "centroid" in case the parallelepiped has spatially constant density. For the case of (iib), they do not coincide for $k_x = k_y = k_z$. In special case of case, $k_x = k_y = k_z = 0$, i.e., the density of the parallelepiped in all direction is uniform and equals $\{\rho_m - (\rho_m - \rho_f)\emptyset_0\}$. For such a homogeneous solid, $[G_{r1}]$ is $[0.5r_1]$ from expression 8 (expression 9). This point is the 'centroid' of the homogeneous parallelepiped, as expected. This confirms the derivations to be correct.

Note in expressions 5 and 8, the location of the C.G. in a sediment/rock body does not have simple linear relationship with densities of the pore fluid (ρ_f) and the matrix (ρ_m), porosity at the surface ($\emptyset_0$), compaction constant (b^{-1}), dimension of the rock/sediment body (x_1, y_1 and z_1) and nor even with the linear density gradients (k_x, k_y and k_z). Second, in case for some terrain, different (empirical) equations are established for (A) density variation in three directions, (B) porosity reduction depth-wise, or if the rock body is irregular shaped, the present work provides just a guideline how to utilize them to locate the C.G. of the rock body. Equations (11) and (13), for example, will be important to use appropriately. Lerche and O'Brien (1987) reviewed several cases of linear reduction of porosity with depth:

$$\emptyset_z = \emptyset_0 - cz. \quad (27)$$

Here 'c' is a constant, which can be location-specific, and 'z' is the depth. In this case, modify Eq. (7) as

$$\rho_{bw}(0,0,z) = \rho_{bwz} = \rho_m - (\rho_m - \rho_f)(\emptyset_0 - cz). \quad (28)$$

This is a linear relation between ρ_{bw} and z, and therefore can be also expressed as Eq. (18), where k_z would be equivalent to $(\rho_m - \rho_f)$. Such changes will certainly affect the C.G.'s location. 'k_z' can be depth dependent (Maxant 1975). In such cases, the presented models would require some modifications. Also, ρ_m and ρ_f can be depth (pressure and temperature) dependent (Djomani et al. 2001). ρ_f can increase depth-wise (Patwardhan 2012). A refined model of C.G. considering these spatial variations could further be attempted. Third, even if the rock/sediment body remains intact, a possible change in 'ρ_f' would ideally shift the C.G. Lower the C.G. locates more stable would be the configuration of the rock/sediment body temporally. In case multi-layered rocks have known modes of variation of density for individual layers, case (ii) presented above can be combined to case (i) to locate the C.G. Metamorphic rocks usually show sporadic variation of density (Reynolds 2011), or sometimes unknown density distribution (Gorbatsevich et al. 2017), therefore setting equations similar to Eqs. (7), (8), (9) and (18) may prove difficult for them. In case, density at 'i'-direction ($i = x, y, z$) is deciphered to be uniform, Eqs. (7) can be set as $k_i = 0$. In *case* (i), for no variation in density downward, 'b' is to be taken as zero in Eq. (7). Also note that the density of the interstitial fluid is expected to be quite different for sedimentary and metamorphosing rocks.

One can differentiate the z-ordinates of G and G′ with respect to time (t). For the multi-layered *case-i*, only the top layer can undergo either sedimentation or erosion (so set dq_2/dt either > or < 0), but not the bottom layer implying: $dq_1/dt = 0$. For known dq_2/dt magnitudes for a layered stratigraphy, one can find out thus the rate at which the C.G. shifts up or down. The present simple exercises in deducing coordinates of the C.G. would also be useful in tectonic understanding of stability issue.

For a homogeneous solid, $g_r = 0.5r_1$, and coordinate of G simplifies. For a 90° rotation of this homogeneous solid, G': $[-0.5z_1, g_y, -0.5x_1]$ (expression 10). *For a homogeneous cube,* $g_x = g_y = g_z = 0.5x_1 = 0.5y_1 = 0.5z_1$: the co-ordinate simplifies. Further, for a 90° rotation of the cube, i.e. for $\theta = 90°$, G' coordinate greatly simplifies to $[-0.5x, 0.5x, -0.5x]$ (expression 11). These coordinates of G' in simplistic cases matches with our intuition, which indicates that the derivations in this article are correct. Relocation of C.G. of a rock body due to rotational faulting (Mukherjee and

Khonsari 2017) can disturb isostatic balance that can be studied further.

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Appendix

The minor derivation in Mukherjee (2017a) is redone

Consider a porous sedimentary layer, and that the pore space is filled up by a single phase (Eq. 2–1 in pp. 32 of Rieke and Chilinganian 1974):

$$\rho_{bw} = \rho_m - \left(\rho_m - \rho_f\right)\emptyset, \tag{29}$$

where ρ_{bw} is the wet bulk density of sediments or the rock, ρ_{m} is the matrix or mineral or grain density, ρ_{f} is the fluid density, and Ø is the porosity expressed as a fraction. If Athy's (1930) empirical law works for the porosity reduction with depth in an exponential manner

$$\emptyset_z = \emptyset_0 \mathrm{e}^{-\mathrm{b}z}, \tag{30}$$

where $\emptyset_z$ is the porosity at depth z, $\emptyset_{z0}$ is the porosity at surface: $z=0$, e is the exponential series, and b: a constant such that $b^{-1}=\lambda$ is compaction constant.

Using the above two eqns,

$$\rho_{\mathrm{bw}} = \rho_{\mathrm{m}} - (\rho_{\mathrm{m}} - \rho_{\mathrm{f}})\emptyset_0 e^{-\mathrm{b}z}. \tag{31}$$

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